Pioneer Journal of Advances in Applied Mathematics
Volume 6, Numbers 1-2, 2012, Pages 25-44
This paper is available online at http://www.pspchv.com/content_PJAAM.html

A FINITE ELEMENT METHOD AND ITS CONVERGENCE FOR AN ELLIPTIC INTERFACE PROBLEM

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Abstract

In this paper, we employ regularity assumptions on the true solution of an elliptic interface problem as well as domain approximation technique made popular by B. Deka [Finite element method with numerical quadrature for elliptic problems with smooth interfaces, J. Comput. Appl. Math. 234(2) (2010), 605-612] and Chen and Zou ([Finite element methods and their convergence for elliptic and parabolic interface problems, Numer. Math. 79(2) (1998), 175-202] and references therein) respectively in the finite element method for elliptic problems with smooth interfaces. It is shown that the discrete solution converges to the exact solution optimally in the order of estimates on L^2-norm and H^1-norm where the regularity of the solution may be different throughout the whole domain. An example is furnished to illustrate the principle.

Received December 24, 2012

2010 Mathematics Subject Classification: 65N15, 65N30, 35K06.

Keywords and phrases: elliptic interface problem, finite element method, optimal error estimate, numerical quadrature.

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1. Introduction

There has been considerable interest in the past four decades on the study of finite element methods for solving elliptic interface problems which are encountered in material sciences, fluid dynamics and stationary heat problems. One of such cases is when two distinct materials or fluids with different conductivities or densities or diffusions are involved. These problems are defined using some approximation hypotheses.

As an early proponent, Babuška [2] studied the first class of finite element method for elliptic interface problem on a smooth domain having a smooth interface. He formulated the problem as a minimization problem and obtained H^1-norm estimate under some approximation assumptions. Bramble and King [5] considered a finite element method in which the domains Ω_1 and Ω_2 are replaced by polygonal domains $\Omega_{1,h}$ and $\Omega_{2,h}$ respectively. This facilitates the transfer of the Dirichlet data and the interface to the polygonal boundaries.

Closely related ideas contained in the works of Huang and Zou [8] and Barrett and Elliot [3] were used via mesh refinement, mortar element and penalized parameter to solve second order elliptic interface problems. However, the results of these earlier mentioned authors lack optimal order of convergence. These limitations motivated other researchers into investigating problems involving more practical regularity assumptions on the exact solutions. Hence in the succeeding decades, the convergence of finite element method by practical regularity assumptions on the exact solution of elliptic interface problems emerged in the works of Chen and Zou [6], Sinha and Deka [12] and others.

Their approach involved domain approximation and has the added advantage that the calculation of stiffness matrix and interface integral related to the jumps of normal derivatives are quite simpler and more practical. In this direction, B. Deka [4] extended the technique by studying the effect of numerical quadrature on elliptic problems with smooth interfaces. One of the hallmarks of his work [13] is in showing the existence of optimal order error estimates on L^2 and H^1-norms for the case where the regularity of the solution is low on whole domain.

In this work, we are concerned with the use of well-known quadrature schemes to evaluate the integral which appears in finite element approximation of an elliptic problem with smooth interface of the form:

$$-\nabla \cdot (a_1 \nabla u) + a_0 u = f \text{ in } \Omega, \tag{1.1}$$

$$u(x) = 0 \text{ on } \partial\Omega, \tag{1.2}$$

$$[u] = 0, \quad \left[\beta \frac{\partial u}{\partial \eta}\right] = j(x) \text{ along } \Gamma, \tag{1.3}$$

where Ω is a convex polygonal domain in $\mathbb{R}^2$ and $\Omega_1 \subset \Omega$ is a region with C^2 boundary $\Gamma = \partial\Omega_1$ and $\Omega_2 = \Omega/\Omega_1$. The functions a_1 and a_0 are assumed to be positive and piecewise constant, i.e., $a_1(x) = a_{1i}$, $a_0(x) = a_{0i}$ for $x \in \Omega_i$, $i = 1, 2$. We show that this replacement will not affect the order of convergence of the technique discussed in [4]. In this regard an improved optimal L^2 and H^1-norm error estimates are derived. This result complements and improves earlier results obtained by B. Deka and other contributors.

The paper is structured as follows: Section 2 introduces basic notions, recalls the quadrature schemes on finite element method from [4] for elliptic interface problem. Section 3 introduces the finite element discretization and some known results for elliptic interface problems. Section 4 discusses the effect of numerical quadrature on finite element technique. The concluding section presents two examples to illustrate the principle.

2. Preliminary Notes

In this paper, the standard notations of Sobolev spaces and norms are used. For $m > 0$ and $1 \le p < \infty$, $W^{m,p}(\Omega)$ denotes Sobolev space of order m with norms

$$\| u \|_{W}^{m,p}(\Omega) = \left(\sum_{0 \le |\alpha| \le m} \| D^{\alpha} u \|_{L^p(\Omega)}^p \right)^{1/p}, \quad 1 \le p < \infty$$

$$\| u \|_{W}^{m,\infty}(\Omega) = \max_{0 \le |\alpha| \le m} \| D^{\alpha} u \|_{L^{\infty}(\Omega)}, \quad p = \infty$$

and for $p = 2$, we write

$$W^{m,2}(\Omega) = H^m(\Omega).$$

$H_0^m(\Omega)$ is a closed subspace of $H^m(\Omega)$, which is the closure of $C_0^{\infty}(\Omega)$ with respect to the norm of $H^m(\Omega)$. For a fractional number s, the Sobolev space is defined as in [1]. The following space

$$X = H^1(\Omega) \cap H^2(\Omega_1) \cap H^2(\Omega_2)$$

equipped with the norm

$$\| v \|_X = \| v \|_{H^1(\Omega)} + \| v \|_{H^2(\Omega_1)} + \| v \|_{H^2(\Omega_2)}$$

is employed in the work. We introduce the weak formulation of the problem by defining the bilinear form $A(\cdots): H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ by

$$A(u, v) = \int_\Omega \{a_1 \nabla u \cdot \nabla v + a(x)uv\}\, dx, \quad \forall u, v \in H^1(\Omega). \tag{2.1}$$

In this paper, the generic positive constant C is always independent of the finite element mesh parameter h.

Variational formulation

The weak formulation (2.1) of the problem consists in multiplying both sides of problem (1.1)-(1.3) by a test function $v \in C_0^\infty(\Omega)$, i.e.,

$$(-(\nabla \cdot a_1 \nabla u) + a_0 u)v = fv \quad \text{in } \Omega$$

$$-(\nabla \cdot a_1 \nabla u)v + a_0 uv = fv.$$

Then integrating by parts over Ω gives

$$\int_\Omega a_1 \nabla u \cdot \nabla v + \int_\Omega a_0 uv = \int_\Omega fv. \tag{2.2}$$

Applying Green's identity to L. H. S. of (2.2), yields

$$\int_\Omega a_1 \nabla u \cdot \nabla v = \int_\Omega a_1 \nabla u \nabla v + \int_\Omega a_0 uv + \int_{\partial\Omega} v \frac{\partial u}{\partial n}. \tag{2.3}$$

Next substituting (2.3) into (2.2), we get

$$\int_\Omega a_1 \nabla u \nabla v + \int_{\partial\Omega} v \frac{\partial u}{\partial n} + \int_\Omega a_0 uv = \int_\Omega fv.$$

(since $v = 0$ on $\partial\Omega$), we obtain

$$\therefore \quad \int_{\Omega} a_1 \nabla u \nabla v + \int_{\Omega} a_0 uv = \int_{\Omega} fv.$$

Employing the interface condition, we get

$$\int_{\Omega} a_1 \nabla u \nabla v + \int_{\Omega} a_0 uv = \int_{\Omega} fv + \int_{\Gamma} jv \quad \forall v \in H_0^1(\Omega)$$

from whence we have the variational problem: find $u \in H_0^1(\Omega)$ satisfying

$$\therefore A(u, v) = (f, v) + \langle j, v \rangle_{\Gamma}, \tag{2.4}$$

where

$$A(u, v) = \int_{\Omega} a_1 \nabla u \nabla v + \int_{\Omega} a_0 uv, \quad (f, v) = \int_{\Omega} fv \text{ and } \langle j, v \rangle_{\Gamma} = \int_{\Gamma} jv.$$

The following result is obtained concerning problem (2.4)

Regularity Result

Theorem 1.1 (See [6, p. 4]). *Let* $f \in L^2(\Omega)$ *and* $j \in H^{1/2}(\Gamma)$. *Then problem* (1.1)-(1.3) *has a unique solution* $u \in X \cap H_0'(\Omega)$ *satisfying the a priori estimate*

$$\| u \|_X \leq C(\| f \|_{L^2(\Omega)} + \| j \|_{H^{1/2}(\Gamma)}).$$

Proof. [See [4, p. 4] The result is well known for the case when $j = 0$.

The general case $j \in H^{1/2}(\Omega)$, where Ω is of class C^2, is proved by finding a function $\widetilde{u} \in X \cap H_0^1(\Omega)$ satisfying

$$[\widetilde{u}] = 0, \; \left[\beta \frac{\partial \widetilde{u}}{\partial n}\right] = j \text{ on } \Gamma$$

and

$$\| \widetilde{u} \|_X \leq C \| j \|_{H^{1/2}(\Gamma)}.$$

Then the result of Theorem 1.1 follows by observing that $v = u - \widetilde{u}$ solves the problem with f replaced with $f + \nabla \cdot (a_1 \nabla \widetilde{u}) + a_0 \widetilde{u}$ and $j = 0$.

3. Finite Element Discretization

Finite element approximation of the problem (1.1)-(1.3) consists of the triangulation τ_i of Ω which is described thus: The domain Ω_1 is approximated by a domain Ω_1^i with the polygonal boundary Γ_i with vertices all lying on the interface Γ. The domain Ω_2 is approximated by Ω_2^i with polygonal exterior and interior boundaries as $\partial\Omega$ and Γ_i respectively. Hence, the triangulation τ_i of the domain Ω satisfies the conditions stated here under (as defined in p. 607 of [4])

(i) $\overline{\Omega} = \bigcup_{k \in \tau_i} K.$

(ii) If $k_1, k_2 \in \tau_i$ and $k_1 \neq k_2$, then either $k_1 \cap k_2 = \varnothing$ or $k_1 \cap k_2$ is a common vertex or edge of both triangles.

(iii) Each triangle $k \in \tau_i$ is either in Ω_1^i or Ω_2^i and intersects Ω in at most two points.

(iv) For each triangle $k \in \tau_i$, let ρ_k, $\overline{\rho}_k$ be the radii of its inscribed and circumscribed circles respectively.

Let $L = \max\{\overline{\rho} : k \in \tau_i\}$. It is assumed that, for some fixed $l_0 > 0$, there exists two positive constants C_0 and C_1 independent of l such that

$$C_0\rho_k \leq l \leq C_1\overline{\rho}_k, \quad \forall k \in \tau_i, \quad \forall l \in (0, l_0).$$

We refer to triangles which have one or two vertices on Γ as interface triangles and denote the set of all such interface triangles by τ_Γ^*. Then, we write

$$\Omega_\Gamma^* = \bigcup_{K \in \tau_\Gamma^*} K.$$

Suppose V_i is a family of finite dimensional subspaces of $H_0^1(\Omega)$ which is defined on τ_i. This family consists of piecewise linear functions vanishing on the boundary $\partial\Omega$. Ciarlet ([7]) gave examples of such finite element spaces. A study of the effect of the numerical quadrature involves defining an approximation of the original bilinear form $A(\cdot, \cdot)$ as follows: For each triangle $K \in \tau_i$, $A_k(x) = A_i$. If $K \subset \Omega_i^i$,

$i = 1, 2$, we define a_i as $a_i(x) = a_k(x), \ \forall K \in \tau_i$.

It therefore follows that the approximation $A_i(\cdot, \cdot): H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ has the definition

$$A_i(u, v) = \sum_{K \in \tau_i} \int_K \{a_k(x) \nabla u \cdot \nabla v + a_0 uv\} \, dx, \quad \forall u, v \in H^1(\Omega).$$

In the case of the inner product, the approximation on V_l and its induced norm are defined by

$$(u, v)_l = \sum_{K \in \tau_i} \left\{ \frac{1}{3} \text{meas}(K) \sum_{i=1}^{3} u(Q_i^K) v(Q_i^K) \right\} \tag{3.1}$$

and

$$\| \psi \|_l = (\psi, \psi)_l^{1/2},$$

where Q^K being the vertices for the triangle K.

The treatment of the interface function j on Ω requires an approximation j_i. This is defined by

$$j_i = \sum_{l=1}^{m_i} j(Q_l) \psi_l \quad \forall j_i \in V_i,$$

with $j \in C(\Gamma)$, where $\{Q_l\}_{l=1}^{m_i}$ is the set of all nodes of the triangulation τ_i lying on the interface Γ and $\{\psi_l^i\}_{l=1}^{m_i}$ is the set of standard nodal basis functions corresponding to $\{Q_l\}_{l=1}^{m_i}$ in the space V_i.

The following Lemma facilitates the application of the analysis on the approximation A_i and the corresponding inner product.

Lemma 3.1 (See [4] p. 608). *The V_i the norms $\| \cdot \|_{L^2(\Omega)}$ and $\| \cdot \|_l$ are equivalent on V_i. Furthermore, we have that for u, $v \in V_l$* and $f \in H^2(\Omega)$,

$$| A_i(u, v) - A(u, v) | \leq Ch \sum_{K \in \tau_\Gamma^*} \| \nabla v_i \|_{L^2(K)} \| \nabla u_i \|_{L^2(K)},$$

$$|(u, v) - (u, v)_i| \leq Ch^2 \| u \|_{H^1(\Omega)} \| v \|_{H^1(\Omega)},$$

$$|(f, v) - (f, v)_i| \leq Ch^2 \| f \|_{H^2(\Omega)} \| v \|_{H^1(\Omega)},$$

$$C \| v \|^2_{H^1(\Omega)} \leq A_i(v, v).$$

To approximate the interface function $j(x)$ by its discrete analogue j_i, we use the next Lemma which describes the accuracy in the theory propounded by Chen and Zou in [6].

Lemma 3.2. *Suppose $j \in H^2(\Gamma)$. Then the following inequality holds:*

$$\left| \int_{\Gamma} j v_i \, ds - \int_{\Gamma_i} j_i v_i \right| \leq Ch^{3/2} \| j \|_{H^2(\Gamma)} \| v_i \|_{H^1(\Omega_\Gamma^*)}, \quad \forall v_i \in V_i,$$

where Ω_Γ^* is the union of all the interface triangles.

The following Lemma is an interface approximation property (see [13]).

Lemma 3.3. *Let $\Omega_\Gamma^* = \bigcup_{k \in \tau_\Gamma^*} K$. Then we have that*

$$\| u \|_{H^1(\Gamma_\Gamma^*)} \leq Ch^{1/2} \| u \|_X.$$

Suppose $\Pi_i : C(\overline{\Omega}) \to V_i$ is the standard interpolation operator corresponding to the space V_i. As the solutions concerned are only in $H^1(\Omega)$ globally, the standard interpolation theory cannot be applied directly. Following the argument of Chen and Zou (see [6, p. 8]), it is possible to obtain optimal error bounds of the interpolant Π_i for $u \in X$ as stated in the following Lemma:

Lemma 3.4 (see [6, p. 8] and [7]). *If $\Pi_i : X \to V_i$ and u are the linear interpolation operator and the solution to the interface problem* (1.1)-(1.3) *respectively, then the following approximation properties hold true:*

$$\| u - \Pi_i u \|_{H^m(\Omega)} \leq Ch^{2-m} \| u \|_X, \quad m = 0, 1$$

hold true.

Proof. For any $u \in X$, suppose u_i be the restriction of u on Ω_i for $i = 1, 2$. Since the interface is of class C^2, the function $u_i \in H^2(\Omega_i)$ can be extended onto the whole of Ω, from whence we obtain the function $\widetilde{u}_i \in H^2(\Omega)$ such that $\widetilde{u}_i = u_i$ on Ω_i and

$$\| \widetilde{u}_i \|_{H^2(\Omega)} \leq C\| u_i \|_{H^2(\Omega_i)} \quad i = 1, 2. \tag{3.2}$$

The existence of such extensions, as obtained in [7] is established as follows: For any triangle $K \in \tau_i \backslash \tau_\Gamma^*$, the standard finite element interpolation theory (cf. [7]) implies that

$$\| \dot{u} - \Pi_i u \|_{H^m(K)} \leq Ch^{2-m} \| u \|_{H^2(K)}, \quad m = 0, 1. \tag{3.3}$$

From standard analysis, we have that $\text{dist}(\Gamma, \Gamma_1) \leq 0(h^3)$. Hence without loss of generality, we may assume that $\text{meas}(K_2) \leq Ch^5$ where $K_i = K \cap \Omega_i$, $i = 1, 2$, for any element $K \in \Gamma_\Gamma^*$. Next using Holder's inequality and $\text{meas}(K_i) \leq Ch^5$, $i = 1, 2$ we obtain

$$\begin{aligned} \| u - \Pi_i u \|_{H^m(K_2)} &\leq Ch^{5(p-2)/4p} \| u - \Pi_i u \|_{W^{m,p}(K_2)} \\ &\leq Ch^{5(p-2)/4p} \| u - \Pi_i u \|_{W^{m,p}(K)} \\ &\leq Ch^{\frac{5(p-2)}{4p}+1-m} \| u \|_{W^{1,p}(K)} \end{aligned} \tag{3.4}$$

for any $p > 2$ and $m = 0, 1$. The standard finite element interpolation theory (see [7]) has been used in the last inequality. Now employing previously defined extensions of $\widetilde{u}_i$ of u_i for $m = 0, 1$ yields

$$\begin{aligned} \| u - \Pi_i u \|_{H^m(K_1)} &= \| \widetilde{u}_1 - \Pi_{u_1} \widetilde{u}_1 \|_{H^m(K_1)} \\ &\leq C \| \widetilde{u}_1 - \Pi_i \widetilde{u}_1 \|_{H^m(K)} \\ &\leq Ch^{2-m} \| \widetilde{u}_1 \|_{H^2(K)} \\ &\leq Ch^{2-m} \| u \|_X. \end{aligned} \tag{3.5}$$

Conclusions of Lemma 3.4 are used in the last inequality. By means of estimates (3.4) and (3.5) we obtain

$$\| u - \Pi_i u \|^2_{H^m(\Omega)}$$

$$\leq Ch^{4-2m} \| u \|^2_X + C \sum_{K \in \tau^*_\Gamma} h^{\frac{5(p-2)}{2p}+2-2m} \| u \|^2_{W^{1,p}(K)}$$

$$\leq Ch^{4-2m} \| u \|^2_X + C \sum_{K \in \tau^*_\Gamma} h^{\frac{5p-10}{2p}+2-2m} \| u \|^2_{W^{1,p}(K)}$$

$$\leq Ch^{4-2m} \| u \|^2_X + C \sum_{K \in \tau^*_\Gamma} h^{\frac{5p-10+4p-4mp}{2p}} \| u \|^2_{W^{1,p}(K)}$$

$$\leq Ch^{4-2m} \| u \|^2_X + C \sum_{K \in \tau^*_\Gamma} h^{\frac{9p-10-4mp}{2p}} \| u \|^2_{W^{1,p}(K)}$$

$$\leq Ch^{4-2m} \| u \|^2_X + C \sum_{K \in \tau^*_\Gamma} h^{\frac{9}{2}-2m-\frac{5}{p}} \| u \|^2_{W^{i,p}(K)}$$

$$\leq Ch^{4-2m} \| u \|^2_X + C \sum_{K \in \tau^*_\Gamma} h^{\frac{9}{2}-2m-\frac{5}{p}} \{\| u \|^2_{W^{1,p}(K_1)} + \| u \|^2_{W^{1,p}(K_2)}\}$$

$$\leq Ch^{4-2m} \| u \|^2_X + C \sum_{K \in \tau^*_\Gamma} h^{\frac{9}{2}-2m-\frac{5}{p}} \{\| \tilde{u} \|^2_{W^{1,p}(K_1)} + \| \tilde{u} \|^2_{W^{1,p}(K_2)}\}. \quad (3.6)$$

By application of Sobolev embedding inequalities for two dimensions (refer to [11] and see that):

$$\| \psi \|_{L^p(\Omega_i)} \leq Cp^{1/2} \| \psi \|_{H^1(\Omega_i)} \quad \forall p > 2, \quad \psi \in H^1(\Omega_i), \quad i = 1, 2, \quad (3.7)$$

from whence we conclude that for $m = 0, 1$ and any $p > 2$ $(p = 10)$ in (3.7)

$$\| \tilde{u}_i \|_{L^{10}(\Omega)} \le \| \tilde{u} \|_{L^{10}(\Omega_i)} \le C \| \tilde{u}_i \|_{H^1(\Omega_i)}$$

$$\| \nabla \tilde{u}_i \|_{L^{10}(K_i)} \le \| \nabla \tilde{u}_i \|_{L^{10}(\Omega_i)} \le C \| \nabla \tilde{u}_i \|_{H^1(\Omega_i)}.$$

In view of the above estimates, it now follows that

$$\| \tilde{u}_i \|_{W^{1,10}(K_i)} \le C \| \tilde{u}_i \|_{H^2(\Omega_i)}.$$

This together with (3.6) gives

$$\| u - \Pi_i u \|^2_{H^m(\Omega)} \le C h^{4-2m} \| u \|^2_X.$$

Then, Lemma 3.4 follows immediately from the estimates (3.3) and (3.8).

4. Error Analysis

The following two results are derived for optimal order error estimates for H^1-norm and L^2-norm respectively.

Theorem 4.1. *Let u, $\hat{u}_l$ be the exact and finite element solutions with quadrature respectively to problem* (2.4). *If $f \in H^2(\Omega)$ and $h \in H^2(\Gamma)$, the following H^1-norm error estimate holds*:

$$\| u - u_l \|_{H^1(\Omega)} \le Ch(\| u \|_X + \| f \|_{H^2(\Omega)} + \| h \|_{H^2(\Gamma)}).$$

Proof. The finite element approximation with quadrature is defined by; find $u_l \in V_l$ such that

$$A_l(u_l, v_l) = (f_l, v_l)_l + (h_l, v_l)_{\Gamma_l} \quad \forall v_l \in V_l, \tag{4.1}$$

where $h_l \in V_l$ and f_l are the linear interpolant of h and f respectively.

Subtracting (2.4) from (4.1), for all $v_l \in V_l$ gives

$$A(u, v_l) - A_l(u_l, v_l)$$

$$= -\{(f_l, v_l)_l - (f_l - v_l)\} - \{(f_l, v_l) - (f_l, v_l)\} - \{\langle h_l, v_l\rangle - \langle h, v_l\rangle_\Gamma\}. \tag{4.2}$$

An appeal to Lemma 3.4 and (4.2), we have that

$$\|\Pi_l u - u_l\|^2_{H^1(\Omega)}$$

$$\leq C\{A_l(\Pi_l u - u, \Pi_l u - u_l) + A_l(u - u_l, \Pi_l u - u_l)\}$$

$$\leq Ch\|u\|_X\|\Pi_l u - u_l\|_{H^1(\Omega)} + \{A_l(u - \Pi_l u, \Pi_l u - u_l)$$

$$- A(u - \Pi_l u, \Pi_l u - u_l)\} + \{A_l(\Pi_l u, \Pi_l u - u_l) - A(\Pi_l u, \Pi_l u - u_l)\}$$

$$+ \{(f_l, u_l - \Pi_l u)_l - (f_l, u_l - \Pi_l u)\} + \{(f_l, u_l - \Pi_l u) - (f, u_l - \Pi_l u)\}$$

$$+ \{\langle h_l, u_l - \Pi_l u\rangle_{\Gamma_l} - \langle h, u_l - \Pi_l u\rangle_\Gamma\}$$

$$\equiv Ch\|u\|_X\|\Pi_l u - u_l\|_{H^1(\Omega)} + I_1 + I_2 + I_3 + I_4 + I_5. \tag{4.3}$$

Using Lemma 3.4, we have that

$$|I_1| \leq Ch\|u\|_X\|\Pi_l u - u_l\|_{H^1(\Omega)}. \tag{4.4}$$

By Lemma 3.1, we get

$$|I_2| \leq Ch\|u\|_X\|\Pi_l u - u_l\|_{H^1(\Omega)} \tag{4.5}$$

and

$$|I_3| \leq Ch^2\|f\|_{H^1(\Omega)}\|\Pi_l u - u_l\|_{H^1(\Omega)}$$

$$\leq Ch^2\|f\|_{H^2(\Omega)}\|\Pi_l u - u_l\|_{H^1(\Omega)}. \tag{4.6}$$

For I_4, we have the following estimate,

$$|I_4| \leq Ch^2\|f\|_{H^2(\Omega)}\|u_l - \Pi_l u\|_{L^2(\Omega)} \tag{4.7}$$

and Lemma 3.2 gives

$$|I_5| \leq Ch^2\|h\|_{H^2(\Gamma)}\|u_l - \Pi_l u\|_{H^1(\Omega)}. \tag{4.8}$$

The result follows by combining (4.3)-(4.8) with Lemma 3.4.

Theorem 4.2. *Let u and u_l be the solutions of the problem* (1.1)-(1.3) *and* (4.1), *respectively. For $f \in H^2(\Omega)$ and $h \in H^2(\Gamma)$, there exists a positive constant C*

independent of h such that

$$\|u-u_l\|_{L^2(\Omega)} \le Ch^2(\|u\|_X + \|f\|_{H^2(\Omega)} + \|h\|_{H^2(\Omega)}).$$

Proof. For the derivation of the L^2-norm from the H^1-norm estimate a standard trick is used. The argument requires the following steps.

(i) The statement of the interface problem, i.e., find $w \in H_0^1(\Omega)$ such that

$$a(w, v) = (u - u_l, v) \quad \forall v \in H_0^1(\Omega). \tag{4.9}$$

(ii) Is the finite element approximations, i.e.,

$$A_l(w_l, v_l) = (u - u_l, v_l) \quad \forall v_l, w_l \in V_l. \tag{4.10}$$

(iii) The elliptic regularity results of the interface problem (4.9), let $w \in X \cap H_0^1(\Omega)$ with jump conditions $[w]$, $\left[\beta_0 \frac{\partial w}{\partial n}\right] = 0$ along Γ then the following a priori estimate

$$\|w\|_X \le C\|u-u_l\|_{L^2(\Omega)} \tag{4.11}$$

hold.

(iv) In relation to the derivation of Theorem 4.1 and using the a priori estimate above, we obtain

$$\| w - w_l \|_{H^1(\Omega)} \le Ch\| u - u_l \|_{L^2(\Omega)} \tag{4.12}$$

and

(v) substituting $v = u - u_l \in H_0^1(\Omega)$ in (4.9) and applying (4.2) gives the following

$$\begin{aligned}
\| u - u_l \|_{L^2(\Omega)}^2 &= A(w, u - u_l) \\
&= A(w - w_l, u - u_l) + A(w_l, u - u_l) \\
&= (w - w_l, u - u_l) + A_l(w_l, u_l) - A(w_l, u_l) \\
&\quad - \{(f_l, w_l)_l - (f_l, w_l)\} - \{(f_l, w_l) - (f, w_l)\}
\end{aligned}$$

$$- \{\langle h_l, w_l \rangle_{\Gamma_l} - \langle h, w_l \rangle_\Gamma\}$$

$$\leq C \| w - w_l \|_{H^1(\Omega)} \| u - u_l \|_{H^1(\Omega)}$$

$$+ Ch \sum_{k \in \tau_\Gamma^*} \| \nabla w_l \|_{L^2(K)} \| \nabla u_h \|_{L^2(K)} - \{(f_l, w_l)_l - (f_l, w_l)\}$$

$$- \{(f_l, w_l) - (f, w_l)\} - \{\langle h_l, w_l \rangle_{\Gamma_l} - \langle h, w_l \rangle_\Gamma\}.$$

In the last inequality Lemma 3.1 is used.

(vi) By Theorem 4.1 and Lemmas 3.1-3.4, we get

$$\| u - u_l \|^2_{L^2(\Omega)}$$

$$\leq C \| w - w_l \|_{H^1(\Omega)} \| u - u_l \|_{H^1(\Omega)}$$

$$+ Ch \sum_{k \in \tau_\Gamma^*} \| \nabla w_l \|_{L^2(K)} \| \nabla u_l \|_{L^2(K)}$$

$$+ Ch^2 (\| f \|_{H^2(\Omega)} + \| h \|_{H^2(\Omega)}) \| w \|_X$$

$$\leq Ch \| w - w_l \|_{H^1(\Omega)} \| u \|_X + Ch \| w - w_l \|_{H^1(\Omega)} \| u - u_l \|_{H^1(\Omega)}$$

$$+ Ch \| u \|_{H^1(\Omega)} \| w - w_l \|_{H^1(\Omega)} + Ch \| u - u_l \|_{H^1(\Omega)} \| w \|_{H^1(\Omega)}$$

$$+ Ch \| u \|_{H^1(\Omega)} \| w \|_{H^1(\Omega_\Gamma^*)} + Ch^2 (\| f \|_{H^2(\Omega)} + \| h \|_{H^2(\Omega)}) \| w \|_X$$

$$\leq Ch \| w - w_l \|_{H^1(\Omega)} \| u \|_X + Ch^2 \| w - w_l \|_{H^1(\Omega)}$$

$$\times (\| f \|_{H^2(\Omega)} + \| h \|_{H^2(\Gamma)}) + Ch \| u \|_{H^1(\Omega)} \| w - w_l \|_{H^1(\Omega)}$$

$$+ Ch^2 (\| f \|_{H^2(\Omega)} + \| h \|_{H^2(\Gamma)}) \| \dot{w} \|_{H^1(\Omega)}$$

$$+ Chh^{1/2} \| w \|_X h^{1/2} \| u \|_X + Ch^2 (\| f \|_{H^2(\Omega)} + \| h \|_{H^2(\Gamma)}) \| w \|_X. \quad (4.13)$$

In conclusion, using (4.11) and (4.12), we have

$$\| u - u_l \|_{L^2(\Omega)}^2 \leq Ch^2 (\| u \|_X + \| f \|_{H^2(\Omega)} + \| h \|_{H^2(\Omega)}) \| u - u_l \|_{L^2(\Omega)}.$$

Dividing both sides of the last inequality by $\| u - u_l \|_{L^2(\Omega)}$ gives the following optimal order error estimate in L^2-norm

$$\| u - u_l \|_{L^2(\Omega)} \leq Ch^2 (\| u \|_X + \| f \|_{H^2(\Omega)} + \| h \|_{H^2(\Omega)}).$$

5. Example

We take the rectangle $\Omega = (1.0) \times (0.1)$ as our computational domain. The interface occurs at $x = \frac{1}{2}$ so that $\Omega_1 = \left(0, \frac{1}{2}\right) \times (0, 1)$ and $\Omega_2 = \left(\frac{1}{2}, 1\right) \times (0, 1)$ and the interface $\Gamma = \overline{\Omega}_1 \cap \overline{\Omega}_2$.

Consider the following elliptic boundary-value problem on Ω.

$$-\nabla \cdot (a_i \nabla u_i) + a_{0i} u_i = f_i \text{ in } \Omega_i, \quad i = 1, 2, \tag{5.1}$$

$$u_1 = 0 \text{ on } \partial\Omega \cap \overline{\Omega}, \quad i = 1, 2, \tag{5.2}$$

$$u_1 |_\Gamma = u_2 |_\Gamma,$$

$$((a_1 \nabla u_1 \cdot n_1) + a_{01} u_1) \cdot \underline{n_1} |_\Gamma + ((a_2 \nabla u_2 \cdot n_2) + a_{02} u_2) \cdot \underline{n_2} |_\Gamma = 0, \tag{5.3}$$

where n_i denotes the unit outer normal vector to Ω_i, $i = 1, 2$.

For the exact solution, we choose

$$u_1(x, y) = \sin(\pi x) \sin(\pi y), \quad (x, y) \in \Omega_1$$

and

$$u_2(x, y) = \sin(5\pi x) \sin(\pi y), \quad (x, y) \in \Omega_2.$$

The right-hand sides f_1 and f_2 of (5.1) are determined from the choice of u_1 and u_2 respectively, with $a_1 = \frac{1}{2}$, $a_2 = \frac{1}{13}$, $a_{01} = (\pi)^2$, and $a_{02} = -(\pi)^2$.

In conclusion, we remark that globally continuous, piecewise, linear finite element functions determined by the triangulations of Ω as prescribed in Section 3, were employed. In Table 5.1, L^2-norm and H^1-norm errors for various step size h are shown with the data depicting

$$\| u - u_h \|_{L^2(\Omega)} = O(h^{1.93})$$

and

$$\| u - u_h \|_{H^1(\Omega)} = O(h^{0.99}).$$

Figures 1 to 6 are attached with Figuers 1-4 showing the numerical solutions when $h = \frac{1}{8}$, $h = \frac{1}{16}$, $h = \frac{1}{32}$ and $h = \frac{1}{64}$, respectively. Figures 5 and 6 are the graphs of the error estimates in h with respect to the L^2-norm and H^1-norm.

Numerical result for the problem 5.

Table 5.1

(h_x, h_y)	h	$\| u - u_n \|_{L^2(\Omega)}$	$\| u - u_n \|_{H^1(\Omega)}$
$\left(\frac{\sqrt{2}}{16}, \frac{\sqrt{2}}{16}\right)$	$\frac{1}{8}$	9.2409×10^{-2}	5.52526×10^{-1}
$\left(\frac{\sqrt{2}}{32}, \frac{\sqrt{2}}{32}\right)$	$\frac{1}{16}$	2.4372×10^{-2}	2.80317×10^{-1}
$\left(\frac{\sqrt{2}}{64}, \frac{\sqrt{2}}{64}\right)$	$\frac{1}{32}$	6.1710×10^{-3}	1.40702×10^{-1}
$\left(\frac{\sqrt{2}}{128}, \frac{\sqrt{2}}{128}\right)$	$\frac{1}{64}$	1.5470×10^{-3}	7.04200×10^{-2}

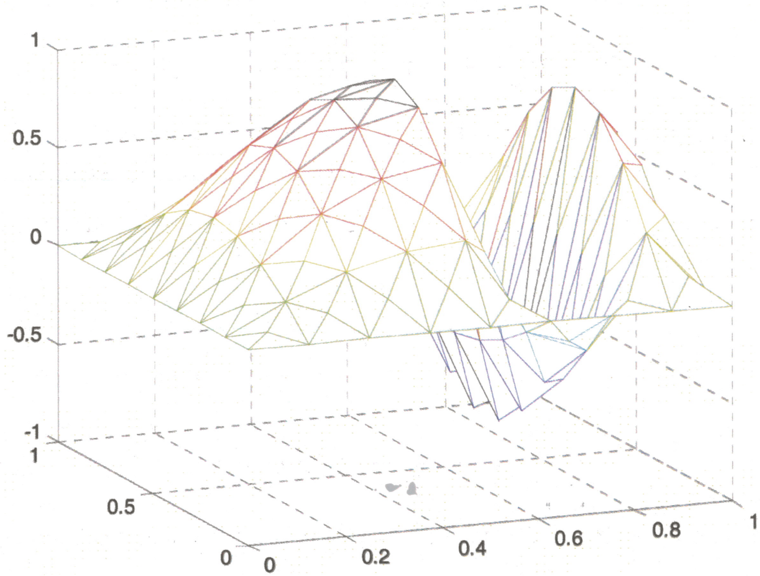

Figure 1. The graph showing the numerical solution when $h = \frac{1}{8}$.

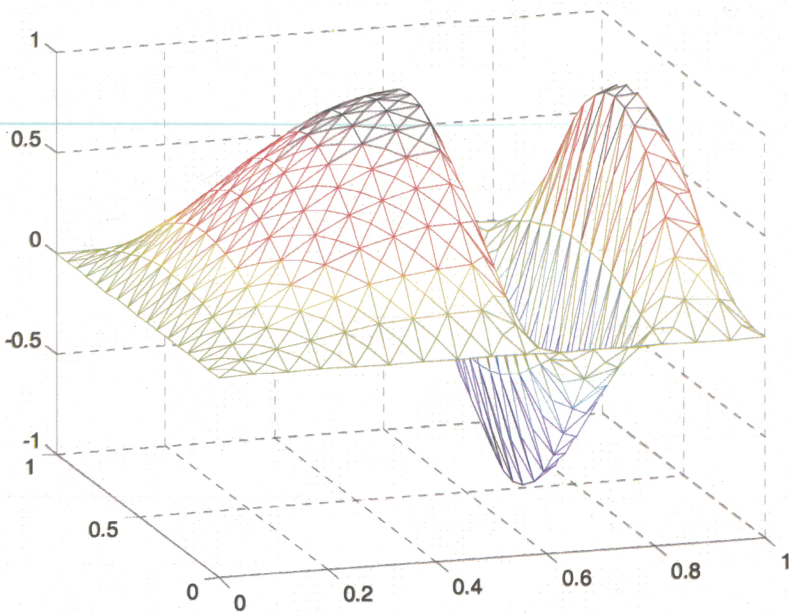

Figure 2. The graph showing the numerical solution when $h = \frac{1}{16}$.

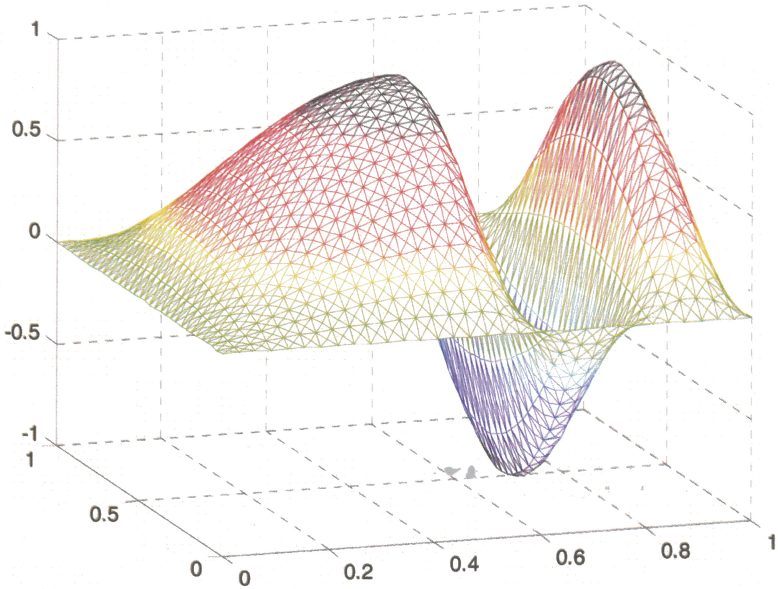

Figure 3. The graph showing the numerical solution when $h = \frac{1}{32}$.

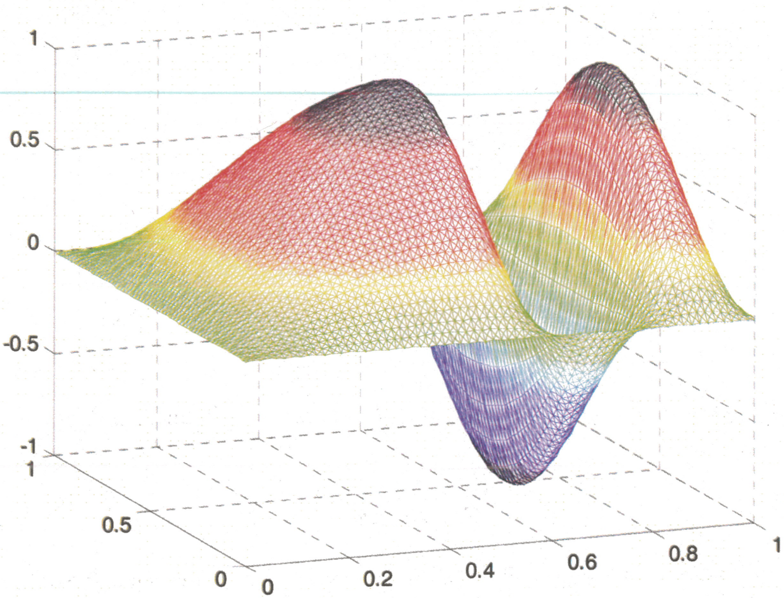

Figure 4. The graph showing the numerical solution when $h = \frac{1}{64}$.

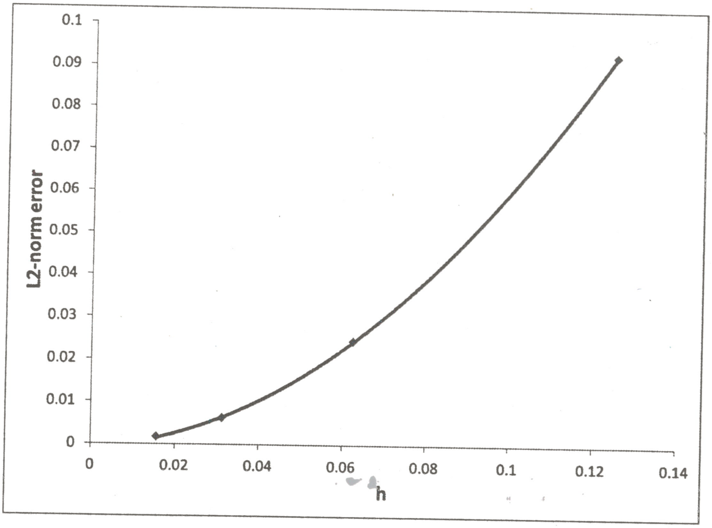

Figure 5. The graph showing that L_2-norm error is quadratic in h.

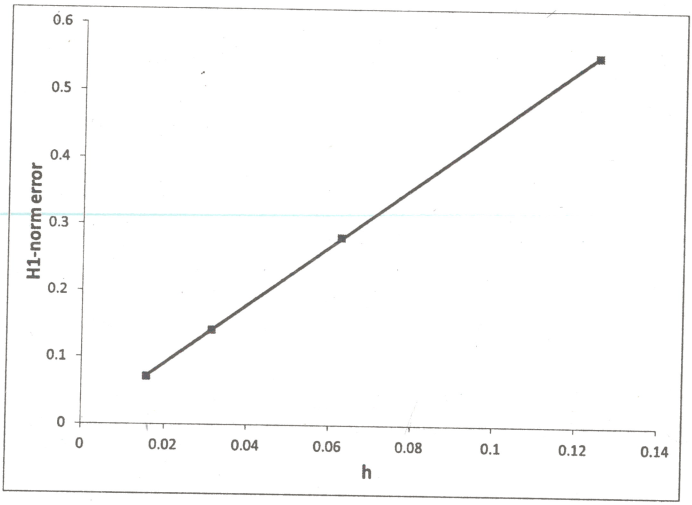

Figure 6. The graph showing that H_1-norm error is linear in h.

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