J. Fixed Point Theory Appl. (2020) 22:98
https://doi.org/10.1007/s11784-020-00834-0
Published online October 29, 2020

Journal of Fixed Point Theory and Applications

A new method for solving split variational inequality problems without co-coerciveness

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Abstract. In solving the split variational inequality problems in real Hilbert spaces, the co-coercive assumption of the underlying operators is usually required and this may limit its usefulness in many applications. To have these operators freed from the usual and restrictive co-coercive assumption, we propose a method for solving the split variational inequality problem in two real Hilbert spaces without the co-coerciveness assumption on the operators. We prove that the proposed method converges strongly to a solution of the problem and give some numerical illustrations of it in comparison with other methods in the literature to support our strong convergence result.

Mathematics Subject Classification. 47H09, 47H10, 49J20, 49J40.

Keywords. Split variational inequality problems, monotone operator, co-coercive, Lipschitz continuous.

1. Introduction

Let C be a nonempty closed and convex subset of a real Hilbert space H and $A : H \to H$ be an operator. The classical Variational Inequality Problem (VIP) for A on C is defined as follows: find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \quad \forall\, x \in C. \tag{1.1}$$

This problem was first introduced by Stampacchia [34] (also independently by Fichera [11]) for modeling problems arising from mechanics. To study the regularity problem for partial differential equations, Stampacchia [34] studied a generalization of the Lax–Milgram theorem and called all problems involving inequalities of such kind, the VIPs, (see also [1,12,20,21]). The VIP (1.1) was later generalized to the following Split Variational Inequality Problem (SVIP) by Censor et al. [9]:

$$\text{Find}\, x^* \in C\, \text{that solves}\, \langle Ax^*, x - x^* \rangle \geq 0 \quad \forall x \in C, \tag{1.2}$$

such that $y^* = Tx^* \in Q$ solves

$$\langle fy^*, y - y^* \rangle \geq 0 \ \forall y \in Q, \tag{1.3}$$

where $A : H_1 \to H_1$, $f : H_2 \to H_2$ are two operators and $T : H_1 \to H_2$ is a bounded linear operator. The SVIP is a special model of the following Split Inverse Problem (SIP):

$$\text{Find } x^* \in X_1 \text{ that solves } IP_1 \tag{1.4}$$

such that

$$y^* = Tx^* \in X_2 \text{ solves } IP_2, \tag{1.5}$$

where X_1 and X_2 are two vector spaces, $T : X_1 \to X_2$ is a bounded linear operator, IP_1 and IP_2 are two inverse problems in X_1 and X_2 respectively (see [5,9]). Note that the first known case of the SIP is the following Split Convex Feasibility Problem (SCFP) introduced and studied by Censor and Elfving [7]:

$$\text{Find } x^* \in C \text{ such that } y^* = Tx^* \in Q. \tag{1.6}$$

Hence, the SVIP (1.2)–(1.3) can also be viewed as an interesting combination of the classical VIP (1.1) and the SCFP (1.6). Thus, it has wide applications in medical treatment of the Intensity-Modulated Radiation Therapy (IMRT), phase retrieval, image reconstruction, signal processing, data compression, among others (for example, see [4,6–9,28,41] and the references therein). Censor et al. [9] proposed and studied the following iterative method for solving SVIP (1.2)–(1.3): for $x_1 \in H_1$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = P_C(I - \lambda A)(x_n + \tau T^*(P_Q(I - \lambda f) - I)Tx_n), n \geq 1, \tag{1.7}$$

where $\tau \in (0, \frac{1}{L})$ with L being the spectral radius of the operator T^*T. They proved that the sequence $\{x_n\}$ generated by (1.7) converges weakly to a solution of (1.2)–(1.3) provided that the solution set of problem (1.2)–(1.3) is nonempty, A, f are L_1, L_2-co-coercive operators and $\lambda \in (0, 2\delta)$, where $\delta := \min\{L_1, L_2\}$.

Since then, other authors have studied the SVIP in Hilbert spaces. See, for example [17,22–24,26]. However, in all of these papers, the convergence of their methods were obtained under the restrictive co-coercive assumption on A and f, thus precluding the use of their methods in many applications. An attempt to overcome this setback was made by Tian and Jiang [40] who proposed the following iterative method: for arbitrary $x_1 \in C$, define the sequence $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ by

$$\begin{cases} y_n = P_C(x_n - \tau_n T^*(I - S)Tx_n), \\ t_n = P_C(y_n - \lambda_n A(y_n)), \\ x_{n+1} = P_C(y_n - \lambda_n A(t_n)), \ n \geq 1, \end{cases} \tag{1.8}$$

where $\{\tau_n\} \subset [a, b]$ for some $a, b \in (0, \frac{1}{||T||^2})$, $\{\lambda_n\} \subset [c, d]$ for some $c, d \in (0, \frac{1}{L})$, $S : H_2 \to H_2$ is a nonexpansive mapping, $T : H_1 \to H_2$ is a bounded linear operator and $A : C \to H_1$ is a monotone and L-Lipschitz continuous mapping. They proved that the sequence generated by Algorithm (1.8) converges weakly to a solution of the following problem: find $x^* \in C$ such that

$$\langle Ax^*, x - x^*\rangle \geq 0 \;\forall x \in C, \text{ and such that } Tx^* \in F(S), \tag{1.9}$$

where $F(S)$ is the set of fixed points of S.
In [41], these authors improved Algorithm (1.8) into the following algorithm to obtain a strong convergent result since strong convergent results are much more desirable in infinite dimensional spaces: for arbitrary $x_1 \in C$, define the sequence $\{x_n\}$, $\{y_n\}$ $\{t_n\}$ and $\{w_n\}$ by

$$\begin{cases} y_n = P_C(x_n - \tau_n T^*(I-S)Tx_n), \\ t_n = P_C(y_n - \lambda_n A(y_n)), \\ w_n = P_C(y_n - \lambda_n A(t_n)), \\ x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n) w_n, \; n \geq 1, \end{cases} \tag{1.10}$$

where $\{\tau_n\}$, $\{\lambda_n\}$, S, T, A are as in Algorithm (1.8), h is a contraction mapping and $\{\alpha_n\} \subset (0,1)$.
Although the underlying operator A in Algorithms (1.8) and (1.10) is freed from the strong co-coercive assumption, but we can see that, even at the expense of too many projections in both algorithms (which may seriously affect the efficiency of these algorithms), these algorithms can only be use to solve the SVIP (1.2)–(1.3) if we set $S = P_Q(I - \lambda f)$ and f is assumed to be co-coercive. Meaning that these methods would still rely on the co-coercive assumption of the second operator f if we intend to use it to solve the SVIP (1.2)–(1.3), which is the problem of interest in this paper.
Based on this, our aim is to design and analyse an iterative method for solving the SVIP (1.2)–(1.3) in two real Hilbert spaces without the restrictive co-coerciveness assumption on the operators A and f usually assumed in many papers (see [17,22–24,26]), and prove that the method converges strongly to a solution of the problem. The strong convergence result is obtained when the operators A and f are monotone and Lipschitz continuous, which is a much more relaxed assumption than the co-coerciveness of the operators. Moreover, as we shall see in Sect. 4, the proof of the strong convergence of our method does not rely on the usual "Two Cases Approach" widely used in many papers to guarantee strong convergence (see for example [17–19,30–32,35–39,41] and the references therein). Furthermore, we give some numerical illustrations of the proposed method in comparison with other methods in the literature to support our strong convergence result.

2. Preliminaries

Let H be a real Hilbert space. Then, an operator $A : H \to H$ is called

- L-co-coercive (or L-inverse strongly monotone), if there exists $L > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq L\|Ax - Ay\|^2 \;\forall\, x, y \in H,$$

- monotone, if

$$\langle Ax - Ay, x - y\rangle \geq \ 0 \ \forall \ x, y \in H,$$

- L-Lipschitz continuous, if there exists a constant $L > 0$ such that

$$||Ax - Ay|| \leq L||x - y|| \ \forall x, y \in H.$$

Clearly, L-co-coercive operators are $\frac{1}{L}$-Lipschitz continuous and monotone but the converse is not always true.

Recall that for a nonempty closed and convex subset C of H, the metric projection denoted as P_C, is a map defined on H onto C which assigns to each $x \in H$, the unique point in C, denoted by $P_C x$ such that

$$||x - P_C x|| = \inf\{||x - y|| : \ y \in C\}.$$

It is well known that P_C is a nonexpansive mapping of H onto C. We also know that the P_C is characterized by the inequality

$$\langle x - P_C x, y - P_C x\rangle \leq 0 \ \forall y \in C.$$

Furthermore, the P_C is known to possess the following property:

$$||P_C x - x||^2 \leq ||x - y||^2 - ||P_C x - y||^2 \ \forall y \in C.$$

For more information and properties of P_C see [13,14].

The following lemmas will be needed in the proofs of our main results.

Lemma 2.1 [10]. *Let H be a real Hilbert space, then for all $x, y \in H$ and $\alpha \in (0, 1)$, the following hold:*

(i) $2\langle x, y\rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2$,
(ii) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$,
(iii) $||x - y||^2 \leq ||x||^2 + 2\langle y, x - y\rangle$.

Lemma 2.2 [33]. *Assume that $A : H \to H$ is a continuous and monotone operator. Then x^* is a solution of (1.1) if and only if x^* is a solution of following problem: find $x^* \in C$ such that*

$$\langle Ax, x - x^*\rangle \geq 0, \ \forall \ x \in C.$$

Theorem 2.3 [15, Theorem 2.3]. *Let $p \in [1, \infty)$ be a rational number except for $p = 1, 2$. Unless $p = np$ for a positive integer n , there is no algorithm which computes the p-norm of a matrix with entries in $\{-1, 0, 1\}$ to relative error with running time polynomial in the dimensions.*

Lemma 2.4 [29]. *Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\alpha_n\}$ be a sequence of real numbers in $(0, 1)$ with condition $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{d_n\}$ be a sequence of real numbers. Assume that*

$$a_{n+1} \leq (1 - \alpha_n)a_n + \alpha_n d_n, \ n \geq 1.$$

If $\limsup_{k\to\infty} d_{n_k} \le 0$ *for every subsequence* $\{a_{n_k}\}$ *of* $\{a_n\}$ *satisfying the condition:*

$$\liminf_{k\to\infty} (a_{n_k+1} - \alpha_{n_k}) \ge 0,$$

then, $\lim_{n\to\infty} a_n = 0$.

3. Proposed method

In this section, we present our proposed method and discuss some motivations for proposing it. We begin with the following assumptions under which our strong convergence is obtained.

Assumption 3.1. Suppose that the following hold:

(a) The feasible sets C and Q are nonempty closed and convex subsets of the real Hilbert spaces H_1 and H_2, respectively.
(b) $A : H_1 \to H_1$ and $f : H_2 \to H_2$ are monotone and Lipschitz continuous with Lipschitz constants L_1 and L_2, respectively.
(c) $T : H_1 \to H_2$ is a bounded linear operator and the solution set $\Gamma := \{z \in VI(A, C) : Tz \in VI(f, Q)\}$ is nonempty, where $VI(A, C)$ is the solution set of the classical VIP (1.1).
(d) $\{\theta_n\} \subset (a, 1 - \alpha_n)$ for some $a > 0$, where $\{\alpha_n\} \subset (0, 1)$.

We next present the proposed method.

Algorithm 3.2. Initialization: *Let* $\tau \ge 0, \lambda \in (0, \frac{1}{L_1}), \mu \in (0, \frac{1}{L_2})$ *and* $x_1 \in H_1$ *be given arbitrary.*
Iterative Steps: *Calculate* x_{n+1} *as follows:*
Step 1. *Set*

$$y_n = P_Q(Tx_n - \mu f T x_n).$$

Compute

$$z_n = Tx_n - \beta_n r_n,$$

where $r_n := Tx_n - y_n - \mu(fTx_n - fy_n)$ *and* $\beta_n := \frac{\langle Tx_n - y_n, r_n\rangle}{||r_n||^2}$, *if* $r_n \neq 0$; *otherwise* $\beta_n = 0$.

Step 2. *Compute*

$$v_n = x_n + \tau_n T^*(z_n - Tx_n),$$

where the stepsize τ_n *is chosen such that for some* $\epsilon > 0$, $\tau_n \in \left(\epsilon, \frac{\|Tx_n - z_n\|^2}{\|T^*(Tx_n - z_n)\|^2} - \epsilon\right)$, *if* $z_n \neq Tx_n$; *otherwise* $\tau_n = \tau$.
Step 3. *Set*

$$u_n = P_C(I - \lambda A)v_n.$$

Compute

$$x_{n+1} = (1 - \theta_n - \alpha_n)v_n + \theta_n w_n,$$

$$w_n = v_n - \gamma_n b_n,$$

where $b_n := v_n - u_n - \lambda(Av_n - Au_n)$ *and* $\gamma_n = \frac{\langle v_n - u_n, b_n\rangle}{||b_n||^2}$, *if* $b_n \neq 0$; *otherwise* $\gamma_n = 0$.
Set $n := n + 1$ *and go back to* **Step 1.**
We now highlight the motivation for the proposed algorithm.

Remark 3.3.
- Observe that Algorithm 3.2 can be viewed as a single projection method for solving the classical VIP in one space H_1 and a single projection method under a bounded linear operator T for solving the second VIP in another space H_2 with no extra projection either on the half-space or on the feasible set. A notable advantage of this method (Algorithm 3.2) for solving SVIP is that the co-coerciveness of the operators A and f usually used in many papers (see for example, [17,22–24,26]) to guarantee convergence, is removed and no extra projection is required under this setting.
- As we shall see in our convergence analysis, the proof of the strong convergence of Algorithm 3.2 (that is, the proof of Theorem 4.3) does not rely on the usual "Two Cases Approach" (Case 1 and Case 2) usually used in numerous papers for solving optimization problems [17,27,30, 31,35–39,41]. Thus, the techniques and ideas employed in our strong convergence analysis are new for solving the problem considered in this paper.
- The choice of the stepsize $\tau_n \in \left(\epsilon, \frac{\|Tx_n - z_n\|^2}{\|T^*(Tx_n - z_n)\|^2} - \epsilon\right)$ used in Algorithm 3.2 does not require priori knowledge of the operator norm $||T||$. Algorithms with stepsize that depends on the operator norm (like in [3,5,9,27,40,41]) require the computation of the norm of the bounded linear operator, which in general is a very difficult task (sometimes impossible) to accomplish as shown in Theorem 2.3.

4. Convergence analysis

Lemma 4.1. *Let* $\{x_n\}$ *be a sequence generated by Algorithm* 3.2. *Then, under Assumption* 3.1, *we have that* $\{x_n\}$ *is bounded.*

Proof. Let $p \in \Gamma$. Since $y_n = P_Q(Tx_n - \mu f Tx_n)$ and $Tp \in VI(f, Q) \subset Q$, then by the characteristics property of P_Q, we obtain that

$$\langle y_n - Tp, y_n - Tx_n + \mu f Tx_n\rangle \leq 0.$$

Thus, by the monotonicity of f, we obtain

$$\begin{aligned}\langle y_n - Tp, r_n\rangle &= \langle y_n - Tp, Tx_n - y_n - \mu f Tx_n\rangle + \mu\langle y_n - Tp, f y_n\rangle \\ &\geq \mu\langle y_n - Tp, f y_n\rangle \\ &= \mu\langle y_n - Tp, f y_n - f Tp\rangle + \mu\langle y_n - Tp, f Tp\rangle \geq 0. \quad (4.1)\end{aligned}$$

From **Step 1** and (4.1), we obtain

$$\begin{aligned}||z_n - Tp||^2 &= ||Tx_n - Tp - \beta_n r_n||^2 \\ &= ||Tx_n - Tp||^2 + \beta_n^2||r_n||^2 - 2\beta_n\langle Tx_n - Tp, r_n\rangle \\ &= ||Tx_n - Tp||^2 + \beta_n^2||r_n||^2 - 2\beta_n\langle Tx_n - y_n, r_n\rangle - 2\beta_n\langle y_n - Tp, r_n\rangle \\ &\leq ||Tx_n - Tp||^2 + \beta_n^2||r_n||^2 - 2\beta_n\langle Tx_n - y_n, r_n\rangle \\ &= ||Tx_n - Tp||^2 + \beta_n^2||r_n||^2 - 2\beta_n^2||r_n||^2 \\ &= ||Tx_n - Tp||^2 - \beta_n^2||r_n||^2. \end{aligned} \tag{4.2}$$

From **Step 2**, (4.2) and Lemma 2.1 (i), we obtain

$$\begin{aligned}||v_n - p||^2 &= ||x_n - p||^2 + \tau_n^2||T^*(z_n - Tx_n)||^2 + 2\tau_n\langle x_n - p, T^*(z_n - Tx_n)\rangle \\ &= ||x_n - p||^2 + \tau_n^2||T^*(z_n - Tx_n)||^2 + 2\tau_n\langle Tx_n - Tp, z_n - Tx_n\rangle \\ &= ||x_n - p||^2 + \tau_n^2||T^*(z_n - Tx_n)||^2 \\ &\quad +\tau_n\left(||z_n - Tp||^2 - ||Tx_n - Tp||^2 - ||z_n - Tx_n||^2\right) \\ &\leq ||x_n - p||^2 + \tau_n^2||T^*(z_n - Tx_n)||^2 - \tau_n||z_n - Tx_n||^2. \end{aligned} \tag{4.3}$$

Thus, by the condition on τ_n, we obtain

$$\begin{aligned}||v_n - p||^2 &\leq ||x_n - p||^2 - \tau_n\left(||z_n - Tx_n||^2 - \tau_n||T^*(z_n - Tx_n)||^2\right) \\ &\leq ||x_n - p||^2. \end{aligned} \tag{4.4}$$

By similar argument used in obtaining (4.2), we get

$$\begin{aligned}||w_n - p||^2 &\leq ||v_n - p||^2 - \gamma_n^2||b_n||^2 \\ &= ||v_n - p||^2 - ||w_n - v_n||^2. \end{aligned} \tag{4.5}$$

Now, observe that

$$\begin{aligned}&||(1-\theta_n-\alpha_n)(v_n - p) + \theta_n(w_n - p)||^2 \\ &\quad= (1-\theta_n-\alpha_n)^2||v_n - p||^2 + \theta_n^2||w_n - p||^2 \\ &\qquad+2(1-\theta_n-\alpha_n)\theta_n\langle v_n - p, w_n - p\rangle \\ &\quad\leq (1-\theta_n-\alpha_n)^2||x_n - p||^2 + \theta_n^2||x_n - p||^2 \\ &\qquad+2(1-\theta_n-\alpha_n)\theta_n||x_n - p||||x_n - p|| \\ &\quad= (1-\alpha_n)^2||x_n - p||^2. \end{aligned} \tag{4.6}$$

Thus, we obtain from **Step 3** that

$$\begin{aligned}||x_{n+1} - p|| &= ||(1-\theta_n-\alpha_n)(v_n - p) + \theta_n(w_n - p) - \alpha_n p|| \\ &\leq ||(1-\theta_n-\alpha_n)(v_n - p) + \theta_n(w_n - p)|| + \alpha_n||p|| \\ &\leq (1-\alpha_n)||x_n - p|| + \alpha_n||p|| \\ &\leq \max\{||x_n - p||, ||p||\} \\ &\;\vdots \\ &\leq \max\{||x_1 - p||, ||p||\}. \end{aligned}$$

Therefore, $\{x_n\}$ is bounded. □

Lemma 4.2. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. If there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges weakly to a point $z \in H_1$ and $\lim_{k\to\infty} \|v_{n_k} - u_{n_k}\| = 0 = \lim_{k\to\infty} \|v_{n_k} - x_{n_k}\|$ for subsequences $\{v_{n_k}\}$ and $\{u_{n_k}\}$ of $\{v_n\}$ and $\{u_n\}$, respectively. Then, $z \in \Gamma$.*

Proof. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$ which converges weakly to some $z \in H_1$. Then, since T is a bounded linear operator, we obtain that $\{Tx_{n_k}\}$ converges weakly to $Tz \in H_2$.

Now, let us assume without loss of generality that $z_n \neq Tx_n$, then $\tau_n \in \left(\epsilon, \frac{||z_n - Tx_n||^2}{||T^*(z_n - Tx_n)||^2} - \epsilon\right)$. Thus, we obtain from (4.3) that

$$\begin{aligned} ||v_n - p||^2 &\leq ||x_n - p||^2 - \tau_n \epsilon ||T^*(z_n - Tx_n)||^2 \\ &\leq ||x_n - p||^2 - \epsilon^2 ||T^*(z_n - Tx_n)||^2, \end{aligned} \tag{4.7}$$

which implies that

$$\begin{aligned} \epsilon^2 ||T^*(z_{n_k} - Tx_{n_k})||^2 &\leq ||x_{n_k} - p||^2 - ||v_{n_k} - p||^2 \\ &\leq ||x_{n_k} - v_{n_k}||^2 + 2||x_{n_k} - v_{n_k}|| ||v_{n_k} - p||. \end{aligned}$$

Thus, by our assumption, we obtain that

$$\lim_{k\to\infty} ||T^*(z_{n_k} - Tx_{n_k})|| = 0. \tag{4.8}$$

Hence, we obtain from (4.3) and (4.8) that

$$\begin{aligned} \tau_{n_k} ||Tx_{n_k} - z_{n_k}||^2 &\leq ||x_{n_k} - p||^2 - ||v_{n_k} - p||^2 + \tau_{n_k}^2 ||T^*(z_{n_k} - Tx_{n_k})||^2 \\ &\leq ||x_{n_k} - v_{n_k}||^2 + 2||x_{n_k} - v_{n_k}|| ||v_{n_k} - p|| \\ &\quad + \tau_{n_k}^2 ||T^*(z_{n_k} - Tx_{n_k})||^2 \to 0, \text{ as } k \to \infty. \end{aligned}$$

Since $0 < \epsilon < \tau_{n_k}$, we obtain that

$$\lim_{k\to\infty} ||Tx_{n_k} - z_{n_k}|| = 0. \tag{4.9}$$

Now, observe that

$$\begin{aligned} \langle Tx_{n_k} - y_{n_k}, r_{n_k} \rangle &= \langle Tx_{n_k} - y_{n_k}, Tx_{n_k} - y_{n_k} - \mu(fTx_{n_k} - fy_{n_k}) \rangle \\ &= ||Tx_{n_k} - y_{n_k}||^2 - \langle Tx_{n_k} - y_{n_k}, \mu(fTx_{n_k} - fy_{n_k}) \rangle \\ &\geq ||Tx_{n_k} - y_{n_k}||^2 - \mu ||Tx_{n_k} - y_{n_k}|| ||fTx_{n_k} - fy_{n_k}|| \\ &\geq (1 - \mu L_2) ||Tx_{n_k} - y_{n_k}||^2, \end{aligned} \tag{4.10}$$

which implies that

$$\begin{aligned} ||Tx_{n_k} - y_{n_k}||^2 &\leq \frac{1}{(1 - \mu L_2)} \langle Tx_{n_k} - y_{n_k}, r_{n_k} \rangle \\ &= \frac{1}{(1 - \mu L_2)} \beta_{n_k} ||r_{n_k}||^2 \\ &= \frac{1}{(1 - \mu L_2)} \beta_{n_k} ||r_{n_k}|| \cdot ||Tx_{n_k} - y_{n_k} - \mu(fTx_{n_k} - fy_{n_k})|| \\ &\leq \frac{1}{(1 - \mu L_2)} \beta_{n_k} ||r_{n_k}|| \left(||Tx_{n_k} - y_{n_k}|| + \mu ||fTx_{n_k} - fy_{n_k}|| \right) \end{aligned}$$

$$\leq \frac{(1+\mu L_2)}{(1-\mu L_2)}||Tx_{n_k}-y_{n_k}||\beta_{n_k}||r_{n_k}|| \\ = \frac{(1+\mu L_2)}{(1-\mu L_2)}||Tx_{n_k}-y_{n_k}||||z_{n_k}-Tx_{n_k}||.$$

Thus, we obtain from (4.9) that

$$||Tx_{n_k}-y_{n_k}|| \leq \frac{(1+\mu L_2)}{(1-\mu L_2)}||z_{n_k}-Tx_{n_k}|| \to 0, \text{ as } k\to\infty. \tag{4.11}$$

Now, by the monotonicity of f and the characteristic property of P_Q, we obtain for all $x \in Q$ that

$$\begin{aligned} 0 &\leq \langle y_{n_k}-Tx_{n_k}+\mu fTx_{n_k}, x-y_{n_k}\rangle \\ &= \langle y_{n_k}-Tx_{n_k}, x-y_{n_k}\rangle + \mu\langle fTx_{n_k}, Tx_{n_k}-y_{n_k}\rangle \\ &\quad + \mu\langle fTx_{n_k}, x-Tx_{n_k}\rangle \\ &\leq \|y_{n_k}-Tx_{n_k}\|\|x-y_{n_k}\| + \mu\|fTx_{n_k}\|\|Tx_{n_k}-y_{n_k}\| \\ &\quad + \mu\langle fTx_{n_k}, x-Tx_{n_k}\rangle \ \forall x\in Q \\ &= \|y_{n_k}-Tx_{n_k}\|\|x-y_{n_k}\| + \mu\|fTx_{n_k}\|\|Tx_{n_k}-y_{n_k}\| \\ &\quad + \mu\left(\langle fTx_{n_k}-fx, x-Tx_{n_k}\rangle + \langle fx, x-Tx_{n_k}\rangle\right) \\ &\leq \|y_{n_k}-Tx_{n_k}\|\|x-y_{n_k}\| + \mu\|fTx_{n_k}\|\|Tx_{n_k}-y_{n_k}\| \\ &\quad + \mu\langle fx, x-Tx_{n_k}\rangle \ \forall x\in Q. \end{aligned} \tag{4.12}$$

Thus, by passing limit as $k\to\infty$, we obtain that

$$\langle fx, x-Tz\rangle \geq 0 \ \forall x\in Q.$$

Therefore, we obtain from Lemma 2.2 that $Tz \in VI(f,Q)$.
On the other hand, we have by our hypothesis that the subsequence $\{v_{n_k}\}$ of $\{v_n\}$ converges weakly to $z\in H_1$. Now, observe that by following similar argument used in obtaining (4.12), we get

$$0 \leq \|u_{n_k}-v_{n_k}\|\|y-u_{n_k}\| + \lambda\|Av_{n_k}\|\|v_{n_k}-u_{n_k}\| + \lambda\langle Ay, y-v_{n_k}\rangle \ \forall y\in C. \tag{4.13}$$

Thus, by our hypothesis, we obtain that

$$\langle Ay, y-z\rangle \geq 0 \ \forall y\in C.$$

Therefore, we obtain from Lemma 2.2 that $z\in VI(A,C)$. Hence, $z\in\Gamma$. □

We now present the main theorem for our strong convergence analysis.

Theorem 4.3. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.2 under Assumption 3.1. If $\lim_{n\to\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n=\infty$. Then, $\{x_n\}$ converges strongly to $p\in\Gamma$, where*

$$||p|| = \min\{||z|| : z\in\Gamma\}.$$

Proof. Let $p \in \Gamma$. Then, we obtain from (4.4) and (4.5) that

$$\begin{aligned}
&||(1-\theta_n)v_n + \theta_n w_n - p||^2 \\
&\quad = ||(1-\theta_n)(v_n - p) + \theta_n(w_n - p)||^2 \\
&\quad = (1-\theta_n)^2||v_n - p|| + \theta_n^2||w_n - p||^2 + 2(1-\theta_n)\theta_n\langle v_n - p, w_n - p\rangle \\
&\quad \le (1-\theta_n)^2||x_n - p|| + \theta_n^2||x_n - p||^2 + 2(1-\theta_n)\theta_n||x_n - p||^2 \\
&\quad = ||x_n - p||^2.
\end{aligned}$$

Thus, from **Step 3**, we obtain

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||(1-\alpha_n)\left[(1-\theta_n)v_n + \theta_n w_n - p\right] - \left[\alpha_n\theta_n(v_n - w_n) + \alpha_n p\right]||^2 \\
&\le (1-\alpha_n)^2||(1-\theta_n)v_n + \theta_n w_n - p||^2 - 2\langle\alpha_n\theta_n(v_n - w_n) \\
&\quad + \alpha_n p, x_{n+1} - p\rangle \\
&\le (1-\alpha_n)||(1-\theta_n)v_n + \theta_n w_n - p||^2 + 2\langle\alpha_n\theta_n(v_n - w_n), p - x_{n+1}\rangle \\
&\quad + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\le (1-\alpha_n)||(1-\theta_n)v_n + \theta_n w_n - p||^2 + 2\alpha_n\theta_n||v_n - w_n|| \cdot ||x_{n+1} - p|| \\
&\quad + 2\alpha_n\langle p, p - x_{n+1}\rangle \\
&\le (1-\alpha_n)||x_n - p||^2 + \alpha_n d_n, \qquad (4.14)
\end{aligned}$$

where $d_n = 2\left(\theta_n||v_n - w_n|| \cdot ||x_{n+1} - p|| + \langle p, p - x_{n+1}\rangle\right)$.
According to Lemma 2.4, to conclude our proof, it suffices to show that $\limsup_{k\to\infty} d_{n_k} \le 0$ for every subsequence $\{||x_{n_k} - p||\}$ of $\{||x_n - p||\}$ satisfying the condition:

$$\liminf_{k\to\infty}\left(||x_{n_k+1} - p|| - ||x_{n_k} - p||\right) \ge 0. \qquad (4.15)$$

To show that $\limsup_{k\to\infty} d_{n_k} \le 0$, suppose that $\{||x_{n_k} - p||\}$ is a subsequence of $\{||x_n - p||\}$ such that (4.15) holds. Then,

$$\begin{aligned}
&\liminf_{k\to\infty}\left(||x_{n_k+1} - p||^2 - ||x_{n_k} - p||^2\right) \\
&\quad = \liminf_{k\to\infty}\left[\left(||x_{n_k+1} - p|| - ||x_{n_k} - p||\right)\left(||x_{n_k+1} - p|| + ||x_{n_k} - p||\right)\right] \\
&\quad \ge 0. \qquad (4.16)
\end{aligned}$$

Now, by **Step 3** and (4.5), we obtain that

$$\begin{aligned}
||x_{n+1} - p||^2 &= ||(1-\theta_n-\alpha_n)(v_n - p) + \theta_n(w_n - p) - \alpha_n p||^2 \\
&= ||(1-\theta_n-\alpha_n)(v_n - p) + \theta_n(w_n - p)||^2 + \alpha_n^2||p||^2 \\
&\quad - 2\alpha_n\langle(1-\theta_n-\alpha_n)(v_n - p) + \theta_n(w_n - p), p\rangle \\
&\le ||(1-\theta_n-\alpha_n)(v_n - p) + \theta_n(w_n - p)||^2 + \alpha_n M \\
&\le (1-\theta_n-\alpha_n)^2||v_n - p||^2 + 2(1-\theta_n-\alpha_n)\theta_n\langle v_n - p, w_n - p\rangle \\
&\quad + \theta_n^2||w_n - p||^2 + \alpha_n M \\
&\le (1-\theta_n-\alpha_n)^2||v_n - p||^2 + \theta_n^2||w_n - p||^2 + \alpha_n M \\
&\quad + (1-\theta_n-\alpha_n)\theta_n||v_n - p||^2 + (1-\theta_n-\alpha_n)\theta_n||w_n - p||^2 \\
&\le (1-\theta_n-\alpha_n)(1-\alpha_n)||v_n - p||^2 \\
&\quad + \theta_n(1-\alpha_n)||w_n - p||^2 + \alpha_n M \\
&\le (1-\theta_n-\alpha_n)(1-\alpha_n)||v_n - p||^2
\end{aligned}$$

$$
\begin{aligned}
&+\theta_n(1-\alpha_n)\left(||v_n-p||^2-||w_n-v_n||^2\right)+\alpha_n M \qquad (4.17)\\
&\le (1-\theta_n-\alpha_n)(1-\alpha_n)||x_n-p||^2+\theta_n(1-\alpha_n)||x_n-p||^2\\
&\quad -\theta_n(1-\alpha_n)||w_n-v_n||^2+\alpha_n M\\
&\le ||x_n-p||^2-\theta_n(1-\alpha_n)||w_n-v_n||^2+\alpha_n M,
\end{aligned}
$$

for some $M>0$. This implies from (4.16) that

$$
\begin{aligned}
\limsup_{k\to\infty}\left[(1-\alpha_{n_k})\theta_{n_k}||w_{n_k}-v_{n_k}||^2\right] &\le \limsup_{k\to\infty}\left[||x_{n_k}-p||^2-||x_{n_k+1}-p||^2+\alpha_{n_k}M\right]\\
&= -\liminf_{k\to\infty}\left[||x_{n_k+1}-p||^2-||x_{n_k}-p||^2\right]\le 0,
\end{aligned}
$$

which gives

$$\lim_{k\to\infty}||w_{n_k}-v_{n_k}||=0. \tag{4.18}$$

Thus, by similar argument used in obtaining (4.11), we get

$$||u_{n_k}-v_{n_k}||\le \frac{(1+\lambda L_1)}{(1-\lambda L_1)}||w_{n_k}-v_{n_k}||\to 0, \text{ as } k\to\infty. \tag{4.19}$$

Combining (4.18) and (4.19), we get

$$\lim_{k\to\infty}||w_{n_k}-u_{n_k}||=0. \tag{4.20}$$

Also, from (4.17) and (4.7), we obtain that

$$
\begin{aligned}
||x_{n_k+1}-p||^2 &\le (1-\theta_{n_k}-\alpha_{n_k})(1-\alpha_{n_k})||v_{n_k}-p||^2\\
&\quad +\theta_{n_k}(1-\alpha_{n_k})||v_{n_k}-p||^2+\alpha_{n_k}M\\
&\le ||v_{n_k}-p||^2+\alpha_{n_k}M\\
&\le ||x_{n_k}-p||^2-\epsilon^2||T^*(z_{n_k}-Tx_{n_k})||^2+\alpha_{n_k}M.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\limsup_{k\to\infty}||T^*(z_{n_k}-Tx_{n_k})||^2 &\le \frac{1}{\epsilon^2}\limsup_{k\to\infty}\left(||x_{n_k}-p||^2-||x_{n_k+1}-p||^2+\alpha_{n_k}M\right)\\
&\le -\frac{1}{\epsilon^2}\liminf_{k\to\infty}\left(||x_{n_k+1}-p||^2-||x_{n_k}-p||^2\right)\le 0,
\end{aligned}
$$

which gives that

$$\lim_{k\to\infty}||T^*(z_{n_k}-Tx_{n_k})||=0. \tag{4.21}$$

Thus, we obtain that

$$\lim_{k\to\infty}||v_{n_k}-x_{n_k}||^2=\tau_{n_k}^2\lim_{k\to\infty}||T^*(z_{n_k}-Tx_{n_k})||=0. \tag{4.22}$$

Also, by (4.18), we obtain that

$$||x_{n_k+1}-v_{n_k}||\le \theta_{n_k}||w_{n_k}-v_{n_k}||+\alpha_{n_k}||v_{n_k}||\to 0, \text{ as } k\to\infty.$$

Thus, we obtain from (4.22) that

$$\lim_{k\to\infty}||x_{n_k+1}-x_{n_k}||=0. \tag{4.23}$$

Since $\{x_{n_k}\}$ is bounded, it follows that there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ that converges weakly to $z \in H_1$ such that

$$\limsup_{k\to\infty} \langle p, p - x_{n_k}\rangle = \lim_{j\to\infty} \langle p, p - x_{n_{k_j}}\rangle = \langle p, p - z\rangle. \tag{4.24}$$

Also, we obtain from (4.19), (4.22) and Lemma 4.2 that $z \in \Gamma$.
Thus, since $p = P_\Gamma 0$, we obtain from (4.24) that

$$\limsup_{k\to\infty} \langle p, p - x_{n_k}\rangle = \langle p, p - z\rangle \leq 0,$$

which implies from (4.23) that

$$\limsup_{k\to\infty} \langle p, p - x_{n_k+1}\rangle \leq 0. \tag{4.25}$$

Using (4.18) and (4.25), we obtain that $\limsup_{k\to\infty} d_{n_k} \leq 0$. Hence, we get that $\lim_{n\to\infty} \|x_n - p\| = 0$. Therefore, $\{x_n\}$ converges strongly to $p = P_\Gamma 0$. □

Remark 4.4. Observe that by setting $H_1 = H_2 = H$, $f = 0$ and $T = I_H$ (the identity operator on H) in Theorem 4.3, we obtain as a corollary, a single projection method (requiring only one projection onto the feasible set C per iteration) for solving the classical VIP (1.1).

5. Numerical examples

We give in this section, some numerical examples (in two infinite dimensional real Hilbert spaces) of Algorithm 3.2 in comparison with Algorithm (1.10) of Tian and Jiang [41], the following Algorithm 5.1 of Pham et al. [27] and Algorithm 5.2 of Reich and Tuyen [28].

Algorithm 5.1. Step 0. *Choose* $\mu_0, \lambda_0 > 0$, $\mu, \lambda \in (0,1)$, $\{\tau_n\} \subset [\underline{\tau}, \bar{\tau}] \subset \left(0, \frac{1}{||T||^2+1}\right)$, $\{\alpha_n\} \subset (0,1)$ *such that* $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$.
Step 1. *Let* $x_1 \in H_1$. *Set* $n = 1$.
Step 2. *Compute*

$$\begin{aligned} u_n &= Tx_n,\\ v_n &= P_Q(u_n - \mu_n f u_n),\\ w_n &= P_{Q_n}(u_n - \mu_n f v_n), \end{aligned}$$

where

$$Q_n = \{w_2 \in H_2 : \langle u_n - \mu_n f u_n - v_n, w_2 - v_n\rangle \leq 0\}$$

and

$$\mu_{n+1} = \begin{cases} \min\left\{\frac{\mu||u_n - v_n||}{||fu_n - fv_n||},\ \mu_n\right\}, & \text{if } fu_n \neq fv_n,\\ \mu_n, & \text{otherwise.} \end{cases}$$

Step 3. *Compute*

$$\begin{aligned} y_n &= x_n + \tau_n T^*(w_n - u_n),\\ z_n &= P_C(y_n - \lambda_n A y_n), \end{aligned}$$

$$t_n = P_{C_n}(y_n - \lambda_n A z_n),$$

where

$$C_n = \{w_1 \in H_1 : \langle y_n - \lambda_n A y_n - z_n, w_1 - z_n\rangle \leq 0\}$$

and

$$\lambda_{n+1} = \begin{cases} \min\left\{\frac{\lambda||y_n - z_n||}{||Ay_n - Az_n||},\ \lambda_n\right\}, & \text{if } Ay_n \neq Az_n, \\ \lambda_n, & \text{otherwise.} \end{cases}$$

Step 4. *Compute*

$$x_{n+1} = \alpha_n x_1 + (1-\alpha_n) t_n.$$

Set $n := n+1$ *and go back to* **Step 2.**

Algorithm 5.2. *For any initial guess* $x_1 = x \in H_1$*, define the sequence* $\{x_n\}$ *by*

$$\begin{aligned} y_n &= VI(C,\ \lambda_n A + I_{H_1} - x_n), \\ z_n &= VI(Q,\ \mu_n f + I_{H_2} - Ty_n), \\ C_n &= \{z \in H_1 : ||y_n - z|| \leq ||x_n - z||\}, \\ D_n &= \{z \in H_1 : ||z_n - Tz|| \leq ||Ty_n - Tz||\}, \\ W_n &= \{z \in H_1 : \langle z - x_n, x_1 - x_n\rangle \leq 0\}, \\ x_{n+1} &= P_{C_n \cap D_n \cap W_n}(x_1),\ n \geq 1, \end{aligned}$$

where I_{H_1} *and* I_{H_2} *are identity operators in* H_1 *and* H_2 *respectively, and* $\{\lambda_n\}$ *and* $\{\mu_n\}$ *are two given sequences of positive numbers satisfying the following condition:*

$$\min\{\inf_n\{\lambda_n\},\ \inf_n\{\mu_n\}\} \geq r > 0.$$

For more details on Algorithms 5.1 and 5.2, see [27, Section 3, Algorithm 1] and [28, Page 12, Section 4.4], respectively.

For the numerical computations, we define

$$\text{TOL}_n := \frac{1}{2}\left(||x_n - P_C(x_n - \lambda A x_n)||^2 + ||Tx_n - P_Q(Tx_n - \mu f T x_n)||^2\right)$$

for Algorithms 3.2, 5.1 and 5.2. While for Algorithm (1.10), we define

$$\text{TOL}_n := \frac{1}{2}\left(||x_n - P_C(x_n - \lambda A x_n)||^2 + ||Tx_n - STx_n||^2\right),$$

and use the stopping criterion $\text{TOL}_n < \varepsilon$ for the iterative processes, where ε is the predetermined error. Note that if $\text{TOL}_n = 0$, then $x_n \in \Gamma$, that is, x_n is a solution of the SVIP considered in this paper.

Note also that, all the codes for the computations are implemented in Matlab 2016 (b). We perform all computations on a personal computer with an Intel(R) Core(TM) i5-2600 CPU at 2.30 GHz and 8.00 Gb-RAM.

Example 5.3. Let $H_1 = H_2 = L_2([0, 2\pi])$ be endowed with inner product

$$\begin{aligned}\langle x, y\rangle &= \int_0^{2\pi} x(t)y(t)\mathrm{d}t \ \forall\ x, y \in L_2([0, 2\pi]) \text{ and norm } ||x|| :\\ &= \Big(\int_0^{2\pi} |x(t)|^2\mathrm{d}t\Big)^{\frac{1}{2}} \ \forall\ x, y \in L_2([0, 2\pi]).\end{aligned}$$

Let $C = \{x \in L_2([0, 2\pi]) : \langle y, x\rangle \leq a\}$, where $y = t + e^{3t}$ and $a = 2$. Then,

$$P_C(x) = \begin{cases} \frac{a-\langle y,x\rangle}{||y||^2_{L_2}} y + x, & \text{if } \langle y, x\rangle > a,\\ x, & \text{if } \langle y, x\rangle \leq a.\end{cases}$$

Also, let $Q = \{x \in L_2([0, 2\pi]) \ : \ ||x - e||_{L_2} \leq b\}$, where $e = t + 2$ and $b = 1$, then Q is a nonempty closed and convex subset of $L_2([0, 2\pi]$. Thus, we define the metric projection P_Q as:

$$P_Q(x) = \begin{cases} x, & \text{if } x \in Q,\\ \frac{x-e}{||x-e||_{L_2}} b + e, & \text{otherwise.}\end{cases}$$

Now, define the operator $A : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ by

$$Ax(t) = \int_0^{2\pi} \Big(x(t) - \Big(\frac{2tse^{t+s}}{e\sqrt{e^2-1}}\Big)\cos x(s)\Big)\mathrm{d}s + \frac{2te^t}{e\sqrt{e^2-1}}, \ x \in L_2([0, 2\pi]).$$

Then A is 2-Lipschitz continuous and monotone on $L_2([0, 2\pi])$ (see [16]). Also define the operator $f : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ by

$$fx(t) = \int_0^t x(s)\mathrm{d}s, \ x \in L_2([0, 2\pi]).$$

Then, f is also Lipschitz continuous and monotone with Lipschitz constant $L_2 = \frac{2}{\pi}$ (see [2]). Let $T : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ be defined by

$$Tx(s) = \int_0^{2\pi} K(s, t)x(t)\mathrm{d}t \ \forall x \in L_2([0, 2\pi]),$$

where K is a continuous real-valued function defined on $[0, 2\pi] \times [0, 2\pi]$. Then T is a bounded linear operator with adjoint

$$T^*x(s) = \int_0^{2\pi} K(t, s)x(t)\mathrm{d}t \ \forall x \in L_2([0, 2\pi]).$$

In particular, we define $K(s, t) = e^{-st}$ for all $s, t \in [0, 2\pi]$.
For Algorithm (1.10), we define the mapping $S : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ by

$$Sx(t) = \int_0^{2\pi} x(t)\mathrm{d}t, \quad x \in [0, 1].$$

Then, S is nonexpansive. We also define $h : L_2([0, 2\pi]) \to L_2([0, 2\pi])$ by

$$hx(t) = \int_0^{2\pi} \frac{1}{2}x(t)\mathrm{d}t, \quad x \in [0, 1].$$

TABLE 1. Numerical results for Example 5.3

Cases		Alg 3.2	Alg (1.10)	Alg 5.1	Alg 5.2
I: ($\varepsilon = 10^{-5}$)	CPU time (s)	2.1401	11.4665	5.9567	5.0631
	No. of Iteration	16	78	39	19
I: ($\varepsilon = 10^{-6}$)	CPU time (s)	2.4087	11.6153	7.1388	6.0725
	No. of Iteration	19	97	47	23
I: ($\varepsilon = 10^{-7}$)	CPU time (s)	2.5032	14.2647	8.2433	6.7110
	No. of Iteration	22	117	55	26
II: ($\varepsilon = 10^{-5}$)	CPU time (s)	1.9927	10.5539	6.4948	5.5345
	No. of Iteration	17	88	42	21
II: ($\varepsilon = 10^{-6}$)	CPU time (s)	2.2628	12.9709	7.6347	6.1992
	No. of Iteration	20	107	50	24
II: ($\varepsilon = 10^{-7}$)	CPU time (s)	2.6799	16.1140	9.0545	7.1845
	No. of Iteration	23	127	58	27
III: ($\varepsilon = 10^{-5}$)	CPU time (s)	2.0941	11.9777	6.8835	5.8391
	No. of Iteration	18	97	45	22
III: ($\varepsilon = 10^{-6}$)	CPU time (s)	2.4939	15.3438	8.0511	6.6407
	No. of Iteration	21	117	53	25
III: ($\varepsilon = 10^{-7}$)	CPU time (s)	2.7110	18.3569	9.3688	7.7041
	No. of Iteration	24	137	61	29

Then, h is a contraction mapping.
Furthermore, we choose $\lambda = \frac{1}{4}$, $\mu = \frac{\pi}{10}$, $\alpha_n = \frac{1}{5n+2}$ and $\theta_n = \frac{1}{2} - \alpha_n$ for all $n \geq 1$. Now, consider the following cases.
Case I: Take $x_1(t) = \sin(2t) + e^{3t}$.
Case II: Take $x_1(t) = 2e^t + t$.
Case III: Take $x_1(t) = t + t^3$.
Using these cases (**Case I–Case III** above), we obtain the numerical results in Table 1 and Figs. 1, 2, 3, which show that our method performs better than Algorithm (1.10) of Tian and Jiang [41], Algorithm 5.1 of Pham et al. [27] and Algorithm 5.2 of Reich and Tuyen [28], in terms of CPU time and number of iteration.

Example 5.4. Let $H_1 = (l_2(\mathbb{R}),\ ||\cdot||_{l_2}) = H_2$, where $l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \dots),\ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ and $||x||_{l_2} := \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$ $\forall x \in l_2(\mathbb{R})$. Now, define the operator $T : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by

$$Tx = \left(0, x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots\right),\ \forall x \in l_2(\mathbb{R}).$$

Then, T is a bounded linear operator on $l_2(\mathbb{R})$ with adjoint

$$T^* y = \left(y_2, \frac{y_3}{2}, \frac{y_4}{3}, \dots\right),\ \forall y \in l_2(\mathbb{R}).$$

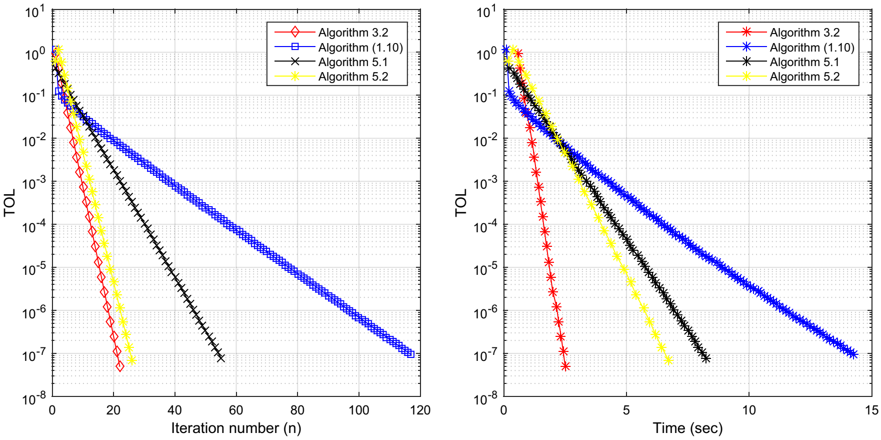

FIGURE 1. The behavior of TOL_n with $\varepsilon = 10^{-7}$ for **Case I** of Example 5.3

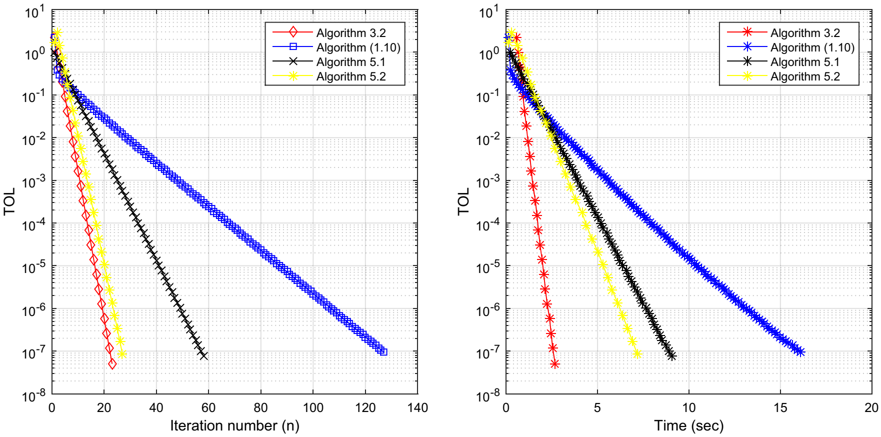

FIGURE 2. The behavior of TOL_n with $\varepsilon = 10^{-7}$ for **Case II** of Example 5.3

To see that T is linear, let $x = (x_1, x_2, x_3, \dots)$, $y = (y_1, y_2, y_3, \dots)$ be arbitrary in $l_2(\mathbb{R})$ and α_1, α_2 be arbitrary in $\mathbb{R}$. Then,

$$\begin{aligned} T(\alpha_1 x + \alpha_2 y) &= \left(0,\ \alpha_1 x_1 + \alpha_2 y_1,\ \frac{\alpha_1 x_2 + \alpha_2 y_2}{2},\ \frac{\alpha_1 x_3 + \alpha_2 y_3}{3}, \cdots\right) \\ &= \left(0,\ \alpha_1 x_1, \frac{\alpha_1 x_2}{2},\ \frac{\alpha_1 x_3}{3}, \cdots\right) + \left(0,\ \alpha_2 y_1, \frac{\alpha_2 y_2}{2},\ \frac{\alpha_2 y_3}{3}, \cdots\right) \\ &= \alpha_1 T(x) + \alpha_2 T(y). \end{aligned}$$

Therefore, T is linear. T is also bounded since $||Tx||_{l_2} \leq ||x||_{l_2}$ $\forall x \in l_2(\mathbb{R})$. The verification that T^* is the adjoint of T follows directly from definition.

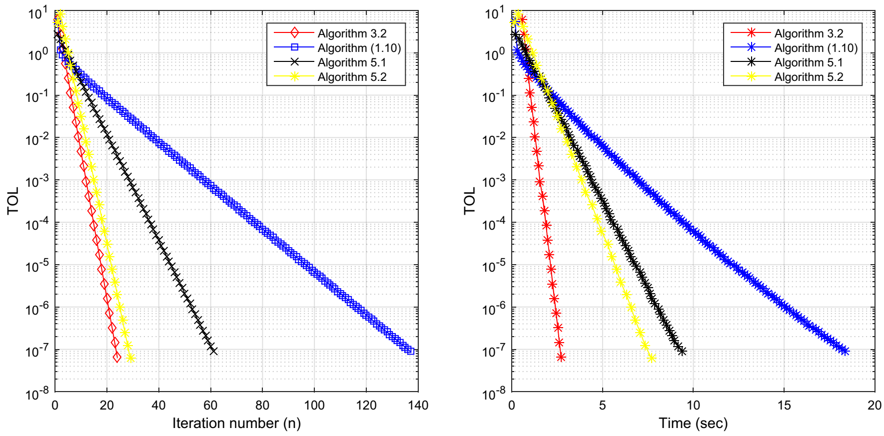

FIGURE 3. The behavior of TOL_n with $\varepsilon = 10^{-7}$ for **Case III** of Example 5.3

TABLE 2. Numerical results for Example 5.4

Cases		Alg 3.2	Alg (1.10)	Alg 5.1	Alg 5.2
A: ($\varepsilon = 10^{-8}$)	CPU time (sec)	0.0150	0.0503	0.0348	0.0346
	No. of Iteration	11	139	49	36
A: ($\varepsilon = 10^{-9}$)	CPU time (sec)	0.0169	0.0523	0.0371	0.0368
	No. of Iteration	13	159	55	41
B: ($\varepsilon = 10^{-8}$)	CPU time (sec)	0.0180	0.0504	0.0401	0.0400
	No. of Iteration	11	132	47	36
B: ($\varepsilon = 10^{-9}$)	CPU time (sec)	0.0181	0.0521	0.0404	0.0402
	No. of Iteration	12	152	53	40
C: ($\varepsilon = 10^{-8}$)	CPU time (sec)	0.0167	0.0505	0.0345	0.0296
	No. of Iteration	11	138	49	21
C: ($\varepsilon = 10^{-9}$)	CPU time (sec)	0.0183	0.0553	0.0429	0.0321
	No. of Iteration	12	158	55	23

We define $C = Q = \{x \in l_2(\mathbb{R}) : ||x-e||_{l_2} \leq b\}$, where $e = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$, $b = 3$ for C and $e = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots)$, $b = 1$ for Q. Then C, Q are nonempty closed and convex subsets of $l_2(\mathbb{R})$. Thus,

$$P_C(x) = P_Q(x) = \begin{cases} x, & \text{if } x \in ||x-e||_{l_2} \leq b, \\ \frac{x-e}{||x-e||_{l_2}} b + e, & \text{otherwise.} \end{cases}$$

Now, define the operators $f, A : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by $Ax = 3x$ and $fx = \frac{8}{3}x$ for all $x \in l_2(\mathbb{R})$.

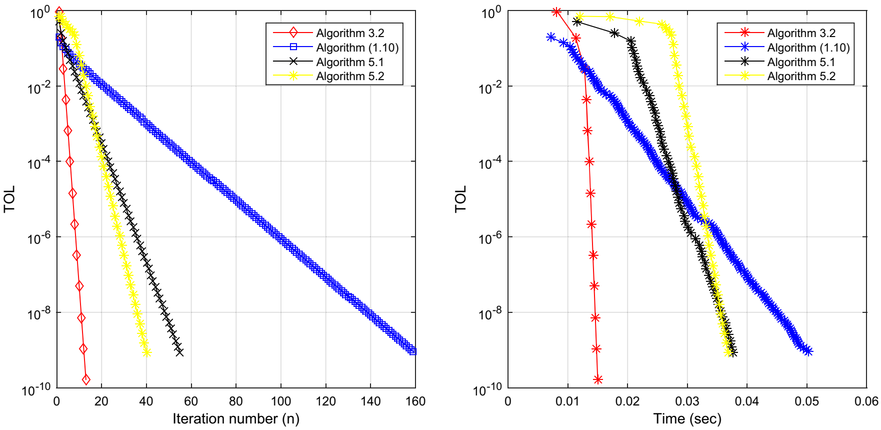

FIGURE 4. The behavior of TOL_n with $\varepsilon = 10^{-9}$ for **Case A** of Example 5.4

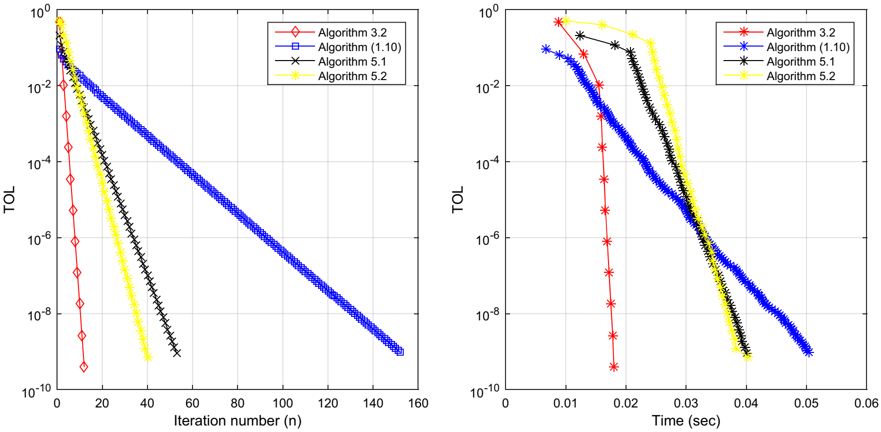

FIGURE 5. The behavior of TOL_n with $\varepsilon = 10^{-9}$ for **Case B** of Example 5.4

More so, for Algorithm (1.10), we define the mappings $S, h : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ by $Sx = (0, x_1, x_2, \dots)$ and $hx = \left(0, \frac{x_1}{2}, \frac{x_2}{2}, \cdots\right)$ for all $x \in l_2(\mathbb{R})$.
Then, we choose $\lambda = \frac{1}{8}$, $\mu = \frac{1}{3}$, $\alpha_n = \frac{1}{5n+2}$ and $\theta_n = \frac{1}{2} - \alpha_n$ for all $n \geq 1$, and consider the following cases.
Case A: Take $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$.
Case B: Take $x_1 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$.
Case C: Take $x_1 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$.
Using (**Case A**–**Case C** above), we obtain the numerical results displayed in Table 2 and Figs. 4, 5, 6, which show that our method still performs better than Algorithm (1.10) of Tian and Jiang [41], Algorithm 5.1 of Pham et al.

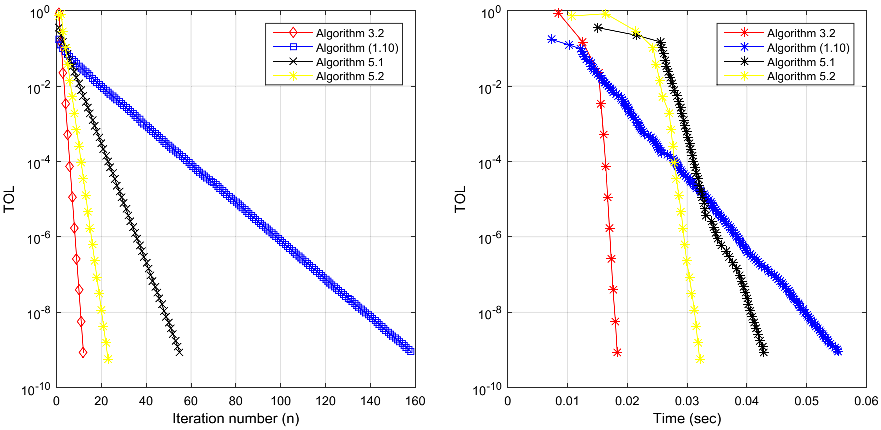

FIGURE 6. The behavior of TOL_n with $\varepsilon = 10^{-9}$ for **Case C** of Example 5.4

[27] and Algorithm 5.2 of Reich and Tuyen [28], in terms of CPU time and number of iteration.

6. Conclusion

Strong convergence of a new iterative method for solving SVIP is established in two real Hilbert spaces under some relaxed assumptions. In particular, the strong convergence result is obtained when the operators A and f are monotone and Lipschitz continuous and this makes our method have much more potential applications than many existing methods for solving the SVIP (1.2)–(1.3). Moreover, the proof of the strong convergence of our method does not rely on the usual "Two Cases Approach" widely used in many papers to guarantee strong convergence. Furthermore, some numerical experiments of this method in comparison with Algorithm (1.10), Algorithms 5.1 and 5.2, are carried out in two infinite dimensional Hilbert spaces. In fact, in all our comparisons, the numerical results demonstrate that our method performs better than these algorithms.

As a concluding remark, we emphasize that the main novelty of this paper is in the design of a method and the proof of its strong convergence to a solution of the SVIP without the restrictive co-coercive assumption on the underlying operators usually assumed in many other existing papers in the literature.

Acknowledgements

The authors sincerely thank the anonymous referees for their careful reading, constructive comments and fruitful suggestions that substantially improved the manuscript. The research of the first author is wholly supported by the

National Research Foundation (NRF) South Africa (S& F-DSI/NRF Free Standing Postdoctoral Fellowship; Grant number: 120784). The first author also acknowledges the financial support from DSI/NRF, South Africa Center of Excellence in Mathematical and Statistical Sciences (CoE-MaSS) Postdoctoral Fellowship. The second author acknowledges the bursary and financial support from Department of Science and Innovation and National Research Foundation, Republic of South Africa Center of Excellence in Mathematical and Statistical Sciences (DSI-NRF COE-MaSS) Doctoral Bursary. The third author is supported by the NRF of South Africa Incentive Funding for Rated Researchers (Grant number 119903). Opinions expressed and conclusions arrived are those of the authors and are not necessarily to be attributed to NRF or CoE-MaSS.

Compliance with Ethical Standards

Competing interests The authors declare that they have no competing interests.

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Accepted: October 15, 2020.