SET THEORY

A mathematical set is a collection of distinct objects normally referred to as elements or members. Sets are usually denoted by a capital letter and the elements by small letters. For example, suppose a company manufactures six different products a, b, c, d, e, and f. Mathematically, we might identify the whole set of products as P and write

$$P = \{a, b, c, d, e, f\}$$

Since e is an element of the set P, we write $e \in P$ and since g is not an element of P, we write $g \notin P$

SET CONCEPTS

1.1. EMPTY OR NULL SET

A set which contains no elements is called an empty or null set and is denoted by ϕ or $\{\}$ Example – the set of rectangles with three sides contains no elements, since there is no rectangle with three sides.

1.2 SUBSET

A subset B, of some set "A" is a set which contains some or all of the elements of A. We write $B \subset A$ which means B is a subset of A. for example, is $A = \{2, 4, 6, 8, 10\}$, $B = \{4, 6, 8\}$ and $X = \{2, 10\}$ then $B \subset A$ and $X \subset A$

Note. Every set is a subset of itself. That is, for all A, $A \subset A$. The empty set is defined to be a subset of every set.

1.3 SET EQUALITY

Two sets are equal only if they have identical elements. Thus, if A = {4, 6, 8} and B = {6, 4, 8,} then A = B but if both sets do not have identical elements then A ≠ B. **Note** that the order of arrangement of the elements do not matter.

1.4 CARDINALITY OF A SET

The cardinality of a set A denoted by n(A) is defined as the number of elements the set A contains. For example. If A = {2, 4, 6, 8, 10} then n(A) = 5.

1.5 THE UNIVERSAL SET

All sets entering into a particular discussion are subsets of a fixed set called the universal set for that discussion. The universal set is the set that contains all the elements under considerations. It is generally denoted by μ or ξ for example . If A = {a, b, c, d,} and B = {b, d, e ,f } then U = {a, b, c, d, e, f} = AUB.

1.6 THE COMPLEMENT OF A SET

If A is any set, with some universal set μ defined, the complement of A, normally written as A' or A^c is defined as all the elements in the universal set that are not contained in set A. For example if we are considering the set of skilled workers S, on a production line, therefore the complement of S i.e. S′ would be all the workers that were not skilled the set of unskilled workers.

(ii) If μ = {a, b, c, d, e, f, g, h} and A { b, c, e, g, h}

then A′ = μ – A = {a, d, f}

1.7 FINITE SET

A set is said to be finite if the elements can be counted and the counting does terminate for example the set of students in Lagos State University.

1.8 INFINITE SET

A set is said to be infinite if the elements or members are uncountable for example, the set of all the positive integers.

1.9 DISJOINT SET

Two sets are disjoint if they have no elements in common. For example if P = { -2, 3, 6, 8} and Q = { -1, 4, 7, 9} then P and Q are disjoint.

1.10 POWER SET (P)

The power set (P) is the number of arrangements of the elements in a set and it is governed by the formula 2^n where n is the number of elements of the set. For example if A = {1,0} and B = {a,b,6}.

Find (i) P (A) (ii) n P(B).

Solution

A = (1, 0)

$\therefore$ $2^n = 2^2 = 4$ where n = 2 is the number of elements in the set.

(i) P(A) = {{1},{0},{1,0}, θ}

B = {a, b, 6}

$\therefore$ $2^n = 2^3 = 8$ (n = 3)

(ii) P(B) = { {a}, {b}, {6}, {a,b}, {b,6}, {a, b,6}, ϕ}

n (P(B)) = 8

1.11 PRODUCT SET

This is an ordered pair of sets in which the first element comes from one set and the other from another set..

For example if A = {5, 7, 9,11} and B = {4, 6, 7, 10}. Find A x B

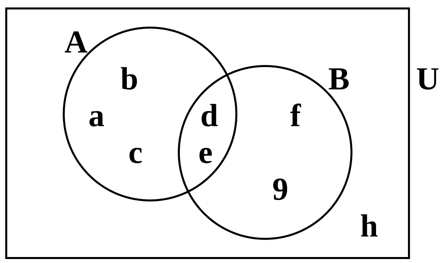
Solution

A x B	4	6	7	10
5	5,4	5,6	5,7	5,10
7	7,4	7,6	7,7	7,10
9	9,4	9,6	9,7	9,10
11	11,4	11,6	11,7	11,10

Thus A x B = [{5,4},{5,6}, 5,7}, {1,10}, {7,4}, {7,6}, {7,7}, {7,10},{9,4},{9,6},{9,7},{9,10} {11,4},{11,6},{11,7}{11,10}]

1.12 VENN DIAGRAM

A Venn diagram is a simple pictorial representation of a set . The relationships between subsets and corresponding universal set can be illustrated by means of Venn diagrams. One of such diagram is shown in the figure below. The rectangle in the diagram represents set U and the subsets. A and B of U are represented by circles drawn
inside the rectangle.



From the figure above

A = {a, b, c, d, e} , B = { d, e f, g} and

U = {a, b, c, d, e, f, g, h}

1.13 OPERATION ON SETS

In ordinary arithmetic and algebra, there are four common operations that can be performed , namely addition, subtraction, multiplication and division. With sets, however, just two operations are common. There are set union and set intersection.

1.13a UNION OF SET

The union of two sets A and B is written as AUB and it is defined as that set which contains all the elements in A or B or both. That is all the elements in A and B without duplication

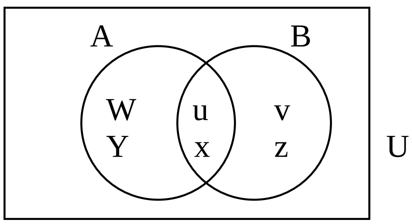
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$n(A \cup B \cup C) = n(A) + n(B) + n(C) - n(A \cap B) - n(A \cap C) - n(B \cap C)$

$+ n(A \cap B \cap C)$

Example If A = {u, w, x, y} and B = {u, v, x, z}

Then $A \cup B$ = {u, v, w, x, y, z}

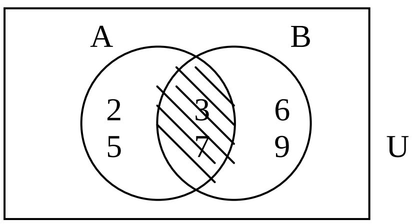


1.13b INTERSECTION OF SETS

The intersection of two sets A and B is written as $A \cap B$ and it is defined as that set which contains all the elements common to both sets A and B

Example

If A = {2, 3, 5, 7} and B = {3, 6, 7, 9} then $A \cap B$ ={3,7}



The shaded portion is $A \cap B$.

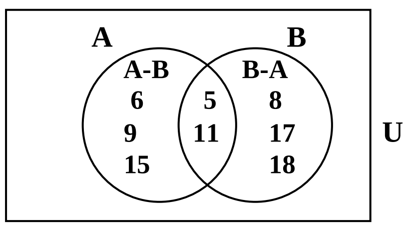
1.14 DIFFERENCE OF A SET

The difference of sets A and B written as A–B, is the set of elements which belong to A but are not in B.

Example

If A = { 5, 6, 9, 11,15} and B = {5, 8,11, 17,18}

Then A-B = {6, 9,15} and B-A = {8, 17, 18}



1.15 LAWS OF OPERATIONS OF UNION, INTERSECTION AND COMPLEMENTATION OF SET

If A, B and C are subset of a universal set U, then the following laws are satisfied

1. Indempotent laws

   (a) $A \cup A = A$

   (b) $A \cap A = A$

2. Association laws

   (a) $(A \cap B) \cup C = A \cup (B \cap C)$

   (b) $((A \cap B) \cap C = A \cap (B \cap C)$

3. Commutative laws

   (a) $A \cap B = B \cap A$

4. Distributive laws

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

5. Identity laws

(a) $A \cup \theta = A$

(b) $A \cap U = A$

(c) $A \cup U = U$

(d) $A \cap \theta = \theta$

6. Complement laws

(a) $A \cup A' = U$

(b) $A \cap A' = \theta$

(c) $(A')' = A$

(d) $U' = \theta$

(e) $\theta' = U$

7. De Morgan laws

(a) $(A \cup A)' = A' \cap B'$

(b) $(A \cap B)' = A' \cup B'$

8. Absorption laws

(a) $A \cup (A \cap B) = A$

(b) $A \cap (A \cup B) = A$

1.16 SET ENUMERATION

Set enumeration involves identifying the number of elements in the various area defined by their union or interaction. There are

(a) 4 district areas for two attribute sets

(b) 8 district areas for three attribute sets

Procedural steps for solving an enumeration problem.

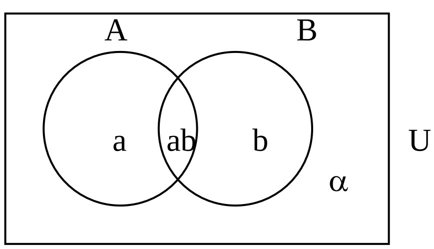
Step 1 ⇒ identify the attribute set

Step 2 ⇒ draw an outline Venn diagram

Step 3 ⇒ use the information given to fill as much of the diagram as possible

Step 4 ⇒ evaluate the number of elements in any unknown area.

NOTATION FOR THE TWO SET PROBLEM



If the two attribute sets in question are labeled as A and B, the Venn diagram in the figure above can be set up.

a = number of elements in set A alone

b = number of elements in set B alone

ab = number of elements in both A and B

α = number of elements in neither A nor B

Thus,

$n(A) = a + ab$

$n(A \cup B) = a + b + ab$

$n(U) = a + b + ab + \alpha$

Example 1.1

A survey was carried out by a local chamber of commerce to discover to what extent computer were being used by firms in the area. 42 firms had both stock control and payroll computerised,75

firms had just one of these two functions computerised and 100 firms had a computerised payroll. If 32 firms had neither of these function computerised.

(a) Find the number of firms that had only payroll computerised

(b) Only stock control computerised

(c) Find the total number of firms included in the survey.

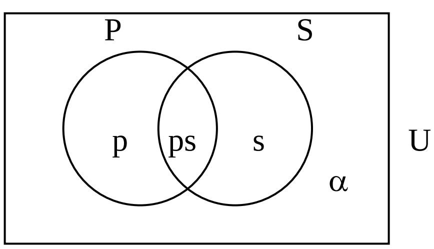
Solution

Step I ⇒ The two attributes are computerised payroll and computerised stock control

Let P = computerised payroll

S = computerized stock control

Step II ⇒ Draw an outline Venn diagram



Step III set up equations from the formation given

Ps = 42 ---------(I)

P + s = 75 --------(ii)

Ps + p = 100 -------(iii)

α = 32 ----------(iv)

step iv ------solve the equation formed above

ps + p = 100 -------equation

42 + p = 100 (from equation ps = 42)

p = 100 – 42 = 58

Substituting p = 58 in equation

58 + s = 75

s = 75 - 58 = 17

(a) The number of firms that had only payroll computerised

p = 58

(b) The number of firms that had only stock control computer

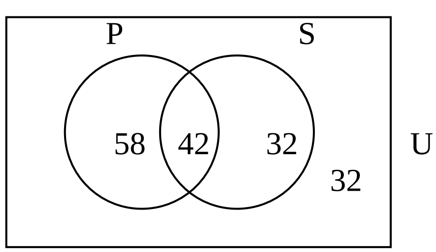
s = 17

(c) The total number of firms included in the survey

= p + s + ps +α

= 58 + 17 + 42 + 32 = 149

The Venn diagram becomes



Example 1.2

In a survey of 290 newspaper readers, 181 of them read the daily times, 142 read the Guardian,117 read the Punch and each read at least one of the three papers. If 75 read the Daily times and the Guardian and 60 read the daily times and Punch and 54 read the Guardian and the Punch

(a) Draw a Venn diagram to illustrate this information

(b) How many readers read

(i) All three papers

(ii) Exactly two of the papers

(iii) Exactly one of the papers

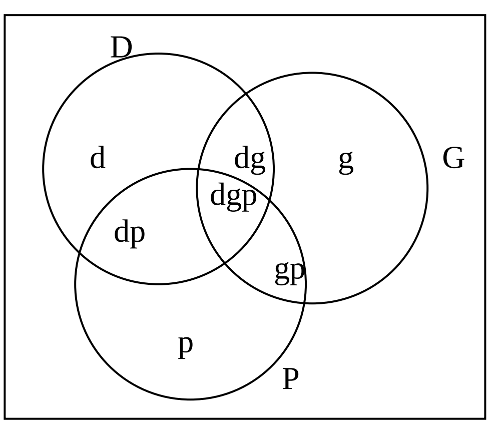
(iv) The Guardian alone?

Solution

(a) Let D = Daily time readers

P = Punch readers

G = Guardian readers



U = 290

Total number of newspaper readers

$d + dp + p + gp + g + dg + dgp = 290$ -------(i)

Daily time readers

$d + dg + dp + dgp = 181$ -----------------(ii)

Punch readers

$p + dp + gp + dgp = 117$ ------------------(iii)

Guardian readers

$g + dg + gp + dgp = 142$ -------------------(iv)

Daily time and Guardian readers

$dg + pdg = 75$ ---------------------(v)

Daily time and Punch readers

$dp + dgp = 60$ ------------(vi)

Guardian and Punch readers

$gp + dgp = 54$ -------------(vii)

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(b) From equation (ii) dgp = number that read all the three papers

$d + dg + dp + dgp = 181$ but $dg + dgp = 75$ (equation)

$\therefore$ $d + dp + 75 = 181$

$d + dp = 181 – 75 = 106$ ------(viii)

From equation (iii)

$p + dp + gp + dgp = 117$ but $dp + dgp = 60$ (equation vi)

$\therefore$

$p + gp + 60 = 117$

$p + gp = 117 - 60 = 57$ ---------------(ix)

from equation (iv)

$g + dg + gp + dgp = 142$ but $gp + dgp = 54$ (equation vii)

$\therefore$ $g + dg + 54 = 142$

$g + dg = 142 – 54 = 88$----------(x)

from equation (i)

(d + dp) + (p +gd) + (g + dg) + dgp = 290

substituting d +dp = 106 , p + gp = 57 and g+ dg = 88 in equation (i) above

We have 106 + 57 + 88 + dgp = 290

251 + dgp = 290

dgp = 290 - 251

dgp   = 39

(i) The number of people that read all the three papers dgp = 39

(ii) From equation (v)

dg + dgp = 75

dg + 39 = 75

dg = 75 – 39 = 36

from equation (vi)

dp + dgp = 60

dp + 39 = 60

dp = 60 – 39 = 21

from equation (vii)

gp +dgp = 54

gp + 39 =54

gp = 54 – 39   =   15

∴ The number of people that read exactly two of the papers

= dg + dp + gp

= 36 + 21 + 15   = 72

(iii) The number of people that read exactly one of the paper

= d + g + p

from equation (viii)

d + dp = 106

d + 21 = 106

d = 106 – 21 = 85

from equation (ix)

18

p + gp = 57

p + 15 = 57

p = 57 – 15 = 42

from equation (x)

g + dg = 88

g + 36 = 88

g = 88 - 36 = 52

∴ d + g + p = 85 + 42 + 52 = 179 the number of people that read exactly one of the papers.

(iv) The number of people that read guardian alone = g = 52

1.18 FURTHER EXAMPLES

Example 1.3 The subsets A, B and C of a universal set are defined as follows

A = {m, a, p, e}

B = {a, e, i, o, u}

C = {L, m, n, o, p, q, r, s, t, u}

List all the element of the following sets

(i) A∪B (ii)A ∪C (iii) A∪ (B∩C) (iv) A – B (v) A∩B

Solution

(i) A∪B = (m,a, p, e, I, o, u)

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(ii) A∪C =(L, m, n, o, p, q ,r, s, t, u, a, e)

(iii) A∪ (B∩C)

B∩C = {o, u}

∴A∪ {B∩C} = {m, a, p, e, o, u}

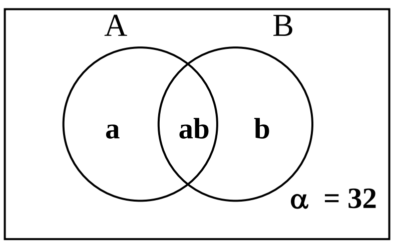
(iv) A – B = {m , p}

(v) A∩B = {a, e}

Example 1.4

2 A manufacturing Company has 237 employees of these 205 participate in at least one of the company's two payroll saving plan, if 176 participate in plan A and 130 participate in plan. B How many participate in both payroll saving plans?.

Solution



U = 237

Given n(A) = 176, n(B) = 130, n(A∪B) = 205

α = 237 - 205 = 32

n (A) = a + ab = 176

n(B) = b + ab = 130

But n(A∪B) = n (A) + n (B) - n (A∩B)

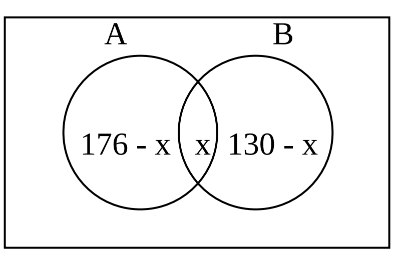
205 = 176 + 130 - ab

205 = 306 - ab

ab = 306 - 205 = 101

∴ The number of employees that participate in both payroll saving plans = 101.

ALTERNATIVELY



U = 237

$n(A \cup B) = 205$

$\therefore\ 176 - x + x + 130 - x = 205$

$306 - x = 205$

$306 - 205 = x$

$101 = x$

$\therefore$ The number of employees that participate in both payroll saving plans = 101

Example 1.5

(a)Show that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(b) show that $(A \cup B)^C \cap A = \theta$

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Solution

(a)To show that $A \cap (B \cup C) = (A \cap B) U (A \cap C)$

Taking L. H. S

Let $X \in A$ and $X \in B \cup C$

$X \in A$ and $X \in B$ or $X \in C$

$X \in A$ and $X \in B$ or $X \in A$ and $X \in C$

$X \in A \cap B$ or $X \in A \cap C$

$X \in (A \cap B) \cup X \in (A \cap C)$

$\therefore\ A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Taking the R.H.S.

$(A \cap B) \cup (A \cap C)$

let $X \in (A \cap B) \cup (A \cap C)$

then $X \in (A \cap B)$ or $(A \cap C)$

$X \in A$ and $X \in B$ or $X \in A$ and $X \in C$

$X \in A$ and $X \in B$ or $X \in C$

Let $X \in A$ and $X \in B \cup C$

$X \in A \cap (B \cup C)$

$\therefore\ (A \cap B) \cup (A \cap C) = A \cap (B \cup C)$

(b)To show that $(A \cup B)^C \cap A = \theta$

Taking L. H. S

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$X \in (A \cup B)^C$ and $X \in A$

$X \in A^C$ and $X \in B^C$ and $X \in A$

$X \notin A$ and $X \in A$ and $X \notin B = \theta$

$\therefore (AVB)^C \cap A = \theta$

Example 1.6

In a market survey 100 traders sell fruits 40 sell apples, 46 oranges, 50 mangoes, 14 apples and oranges, 15 apples and mangoes and 10 sell the three fruit . Each of the traders sells at least one of the three fruits.

(i) Represent the information in a Venn diagram

(ii) Find the number that sell orange and mangoes only

(iii) What percentage of the traders sells two fruits only.

Solution

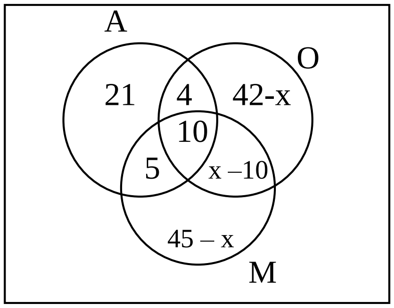
(i) Given $n(U) = 100 = n(A \cup O \cap M)$

Let apples = A , oranges = O and mangoes = M

$n(A) = 40$, $n(O) = 46$, $n(M) = 50$

$n(A \cap O) = 14$, $n(A \cap M) = 15$, $n(A \cap O \cap M) = 10$

Let $n(O \cap M) = x$



U = 100

$n(A \cap M \cap O') = 15 - 10 = 5$

$n(A \cap O \cap M') = 14 - 10 = 4$

$n(O \cap M \cap A') = x - 10$

$n(A \cap M' \cap O') = 40 - (5 + 10 + 4) = 40 - 19 = 21$

$n(O \cap A' \cap M') = 46 - (4 + 10 + x - 10) = 46 - (4 + x)$

$= 46 - 4 - x = 42 - x$

$n(M \cap A' \cap O') = 50 - (5 + 10 + x - 10)$

$= 50 - (5 + x) = 50 - 5 - x = 45 - x$

To solve for x

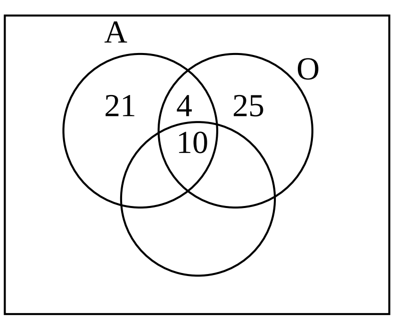
$n(A \cup O \cup M) = 100$

$\therefore\ 21 + 42 - x + 45 - \cancel{x} + 5 + 4 + \cancel{x} \cancel{- 10} + \cancel{10} = 100$

$117 - x = 100$

$117 - 100 = x = 17$

The Venn diagram becomes



5 7

28
M

(i) the number that sell oranges and mangoes only = 7

(ii) The number of trader that sell two fruits only

= 4 + 5 + 7 = 16

$\therefore$ % of traders that sell two fruits only = $\frac{16}{100}$ x 100% = 16%.

5. In a class of 40 students, 25 speak Hausa, 16 speak Igbo, 21 speak Yoruba and each speak at least one of these languages. If 8 speak Hausa and Igbo, 11 speak Hausa and Yoruba and 6 speak Igbo and Yoruba.

(a) Draw a Venn diagram to illustrate this information using x to represent the number of students who speak all three languages .

(b) calculate the value of x.

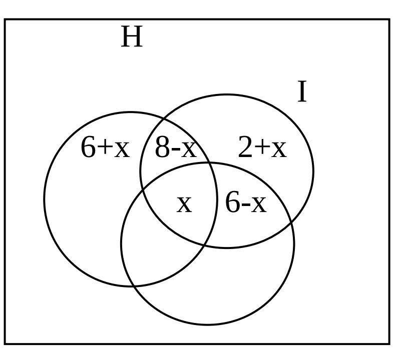
Solution

Given n(U) = 40 = n(H∪Y∪I)

Let Igbo = I, Hausa, = H, Yoruba = Y

n(I) = 16 , n(Y) = 21 n(H) = 25

n(H∩I) = 8 n(H ∩Y) = 11, n(I ∩Y) = 6 n(H∩I∩Y) = x



U = 40

11-x

4 + x Y

$n(H \cap I \cap Y') = 8 - x$

$n(Y \cap I \cap H') = 6 - x$

$n(H \cap Y \cap I') = 11 - x$

$n(H \cap Y \cap I') = 25 - (8 - x + x + 11 - x)$

$= 25 - (19 - x) = 25 - 19 + x = 6 + x$

$n(I \cap Y' \cap H') = 16 - (8 - x + x + 6 - x)$

$= 16 - (14 - x) = 16 - 14 + x = 2 + x$

$(Y \cap H' \cap I') = 21 - (11 - x + x + 6 - x)$

$= 21 - (17 - x) = 21 - 17 + x = 4 + x$

(b) $n(H \cup Y \cup I) = 40$

$\therefore\ 6 + x + 4 + x + 2 + x + 8 - x + 11 - x + 6 - x + x = 40$

$37 + x = 40$

$x = 40 - 37 = 3$

$\therefore\ x = 3$

Example 1.8

A, B and C are subsets of the universal set such that,

U = (0, 1, 2, 3, - - -,12)

A = (x : 0 ≤ x ≤ 7)

B = (4, 6, 8, 10, 12)

C = (1 < y < 8) where y is a prime number.

(a) Draw a Venn diagram to illustrate the information given above

(b) Find (i) $(B \cup C)'$; (ii) $A' \cap B \cap C$

Solution

Given U = { 0, 1,2, 3, 4, 5, 6, 7, 8, 9,10, 11 ,12}

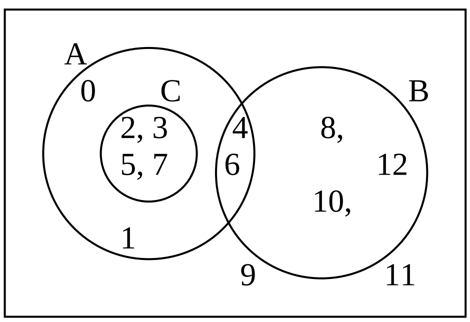
A = { 0, 1, 2, 3,4, 5, 6, 7, }

B = {4, 6, 8, 10, 12}

C = {2, 3, 5, 7}

Note: A prime number is that number that is divisible by itself and 1 only.

(a)



(bi) $B \cup C$ = (2,3,4, 5, 6, 7, 8, 10, 12)

$\therefore (B \cup C)$ = {0, 1, 9, 11}

(ii) A' = {8, 9, 10, 11, 12}

$\therefore A' \cap B \cap C = \theta$

1.19 EXERCISES

1(a) Define the following set concepts

(i) subset (ii) Null set (iii) Union of sets (iv) cardinality of a set

(v) intersection of set

(b) In a class of 400 students, 100 are boys and 150 are girls, 75 are both boys and girls. How many students are neither boys or girls.

2. The sets $A = \{1, 3, 5, 7, 9, 11\}$

$B = \{2, 3, 5, 7, 11, 15\}$ and

$C = \{3, 6, 9, 12, 15\}$ are subsets of

$U = \{1, 2, 3, \ldots, 15\}$

(a) Draw a Venn diagram to illustrate the given information

(b) Use your diagram to find (i) $C \cap A'$ (ii)$A' \cap (B \cup C)$

3 In a group of 120 students, 75 of them play football, 65 play table tennis and 53 play hockey. If 35 of the students play both football and table tennis, 30 play both football and hockey , 21 play both table tennis and hockey and each of the students plays at least one of the three games.

(a) How many of them play

(i) all the three games

(ii) exactly two of the three game

(iii) exactly one of the three games

(iv) football alone?

4. In a class of 65 students, 35 offers Accounts, 25 offers Economics and 25 offers Mathematics each student offer at least one of the three subjects if 12 students offer both Accounts and Economics ,7 students offer both Economic and Mathematics and 15 students offer Account and Mathematics.

(a) Draw a Venn using x to represent the number of students who offer all the three subjects

(b) Find the value of x.

5. A Company, which has 5 regular customers, stocks products r, s, t, u, v, w, x, and y. Customer A buys products r, s, t and v only customers B, C, D and E buy products as represented in set form below

A = {r, s, t, u}, B = {r, t, v, w, x} C = {r, t, x}, D = {r, v, w}

And E = {r, v, w, x} specify the element of each of the following sets giving its meaning in word,

(a) A universal set U (b) $C \cup D$ (c) $E \cap B$ (d) C' (e) $(A \cup C) \cap B'$

(f) $A \cap B \cap C \cap D \cap E$ (g) $(A \cup B \cup C \cup D \cup E)'$

6. A, B and C are three intersecting sets. Given that ac = 15, b = 65, c = 51. abc + ab = 15, a+ b + ab = 117 , b + c + bc = 128

$\alpha = 5$ and n(U) = 200

Calculate (i) abc (ii) a + b + c (iii) n(A) (iv) a (v) ab

7. If U = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}

X = {1, 2, 4, 6, 7,8, 9}

Y = {1,2, 3, 4, 7, 9} and

Z = {2, 3, 4, 7, 9}.

Find (i)x' (ii) z – x (iii) $(x \cap Y)'$ (iv) $n(x \cap Y \cap Z')$

(v) $X' \cup Z'$

8. If P = {1, 2, 3, 4} and Q = {3, 5, 6}. Find

(i) $P \cap Q$ (ii) $P \cup Q$ (iii) $(P \cap Q) \cup Q$ (iv) Q – P (v) $(P \cup Q) \cup P$

9. The subsets P and Q of the universal set U are defined as follows

$X = (x : 30 \leq x \leq 50)$

P = (prime number)

Q = (odd numbers)

List the elements of (i) P (ii) Q (iii) P′ (iv)use set notation to describe the set R = {33,35,39,45,49} in terms of P and Q.

10. There are 70 women in a club. Each plays at least one of the following games, volleyball, basketball and table tennis. 20 plays volleyball only,10 plays basketball only and 6 play table tennis

only. 4 play all three games and equal number play two games only

(a) Illustrate this information in a Venn diagram

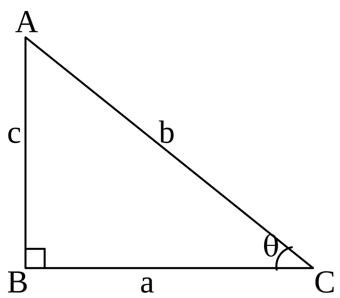
(b) Find the number of women who play volleyball,

TRIGONOMETRY

Trigonometry is the study of angles and sides.

4 .1 The Trigonometry Ratios

Suppose we are given a right- angled triangle ABC where A, B, C are the angles and the sides facing angle A, B, C are a, b, c respectively, then the triangle follows



Let angle c be θ as indicated above then θ = opposite side, a = adjacent side and the side facing the right – angle, b = hypotenuse.

Sine θ = sin θ $= \frac{\text{opposite}}{\text{hypotenuse}} = \frac{c}{b}$

Cosine θ = cos θ $= \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{a}{b}$

Tangent = tan θ $= \frac{\text{opposite}}{\text{adjacent}} = \frac{c}{a}$

Cosecant θ = cosec θ $= \frac{I}{\sin\theta} = \frac{I}{\frac{\text{opposite}}{\text{hypotenuse}}}$

$= \frac{\text{hypotenuse.}}{\text{opposite}} = \frac{b}{c}$

Secant θ = sec θ $= \frac{I}{\sin\theta} = \frac{\text{adjacent}}{\text{hypotenuse.}}$

$= \frac{\text{hypotenuse.}}{\text{adjacent}} = \frac{b}{a}$

Cotangent θ = cot θ $= \frac{I}{\tan\theta} = \frac{I}{\frac{\text{opposite}}{\text{adjacent}}} = \frac{a}{c}$

Note: Tan θ $= \frac{\sin\theta}{\cos\theta}$.

4.2 TRIGONOMETRY IDENTITIES

1. $\sin^2\theta + \cos^2\theta = 1$

   1a. $\sin^2\theta = 1 - \cos^2\theta$

   1b. $\cos^2\theta = 1 - \sin^2\theta$

2. $1 + \tan^2\theta = \sec^2\theta$

3. $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

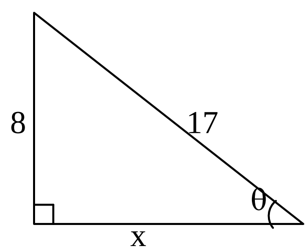
Example 4 .1

Given that $\sin\theta = \frac{8}{17}$, calculate the value of

(a) $\cos\theta$

(b) $\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$

Solution



$\sin\theta = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{8}{17}$

Using Pythagorus theorem $17^2 = 8^2 + x^2$

$\therefore\ 17^2 - 8^2 = x^2$

$\sqrt{289 - 64} = x$

$\sqrt{225} = x$

$x = 15$

(a) $\cos\theta = \dfrac{\text{Adjacent}}{\text{Hypotenuse}} = \dfrac{15}{17}$

(b) $$\frac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta} = \frac{\frac{15}{17} + \frac{8}{17}}{\frac{15}{17} - \frac{8}{17}} = \frac{\frac{23}{17}}{\frac{7}{17}}$$

$$= \frac{23}{17} \text{ x } \frac{7}{17} = \frac{23}{7}$$

ALTERNATIVELY

$\text{Cos}^2\theta = 1 - \sin^2\theta$

$\therefore \text{ Cos } \theta = \sqrt{1 - \sin^2\theta}$

$$= \sqrt{1 - (\frac{8}{17})^2} = \sqrt{\frac{1}{1} - \frac{64}{289}} = \sqrt{\frac{289 - 64}{289}}$$

$$= \sqrt{\frac{225}{289}} = \frac{15}{17}$$

Example 4.2

Simplify, $\cos^2 x (\sec^2 x + \sec^2 x \tan^2 x)$

Solution

$\text{Cos}^2 x (\sec^2 x + \sec^2 x \tan^2 x)$.

$\text{Cos}^2 x \sec^2 x (1 + \tan^2 x)$

$\text{Cos}^2 x \sec^2 x \sec^2 x$

$\cancel{\text{Cos}^2}\mathbf{x} \text{ x } \frac{1}{\cancel{\cos^2 x}} \sec^2\mathbf{x} = \sec^2\mathbf{x}$

Example 4.3

If $\tan\theta = \dfrac{m^2 - n^2}{2mn}$, find sec θ

Solution

$$\text{Tan } \theta = \frac{m^2 - n^2}{2mn}$$

$$\text{Sec}^2\theta = 1 + \tan^2\theta$$

$$\text{Sec } \theta = \sqrt{1 + \tan^2\theta}$$

$$\tan^2\theta = \left(\frac{m^2 - n^2}{2mn}\right)^2 = \left(\frac{m^2 - n^2}{2mn}\right)\left(\frac{m^2 - n^2}{2mn}\right)$$

$$= \frac{m^4 - m^2n^2 - m^2n^2 + n^4}{4m^2n^2}$$

$$= \frac{m^4 - 2m^2n + n^4}{4m^2n^2}$$

$$\therefore \sec\theta = \sqrt{\frac{1}{1} + \frac{m^4 - 2m^2n + n^4}{4m^2n^2}}$$

$$= \sqrt{\frac{4m^2n^2 + m^4 - 2m^2n^2 + n^4}{4m^2n^2}}$$

$$= \sqrt{\frac{m^4 + 2m^2n + n^4}{4m^2n^2}}$$

$$= \sqrt{\frac{(m^2)^2 + 2m^2n^2 + (n^2)^2}{4m^2n^2}}$$

$$= \sqrt{\left(\frac{m^2 + n^2}{2mn}\right)^2}$$

$$\sec\theta = \frac{m^2 + n^2}{2mn}$$

Example 4 .4

Show that $\dfrac{\sin^2 x}{1 - \cos x} + \dfrac{Sin^2 x}{1 + Cos\, x} = 2$

Solution

$$\frac{\sin^2 x}{1 - \cos x} + \frac{\sin^2 x}{1 + \cos x} = 2$$

Taking the L. H. S

$$\frac{\sin^2 x}{1 - \cos x} + \frac{\sin^2 x}{1 + \cos x} = 2$$

$$\frac{\sin^2 x(1+\cos x) + \sin^2 x(1-\cos x)}{(1-\cos x)(1+\cos x)}$$

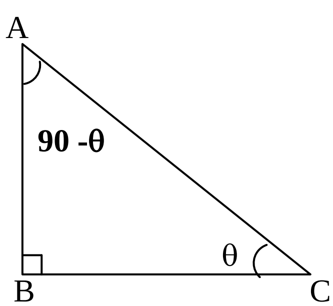
$$\frac{\sin^2 x + \cancel{\sin^2 x \cos x} + \sin^2 x - \cancel{\sin^2 x \cos x}}{1 + \cancel{\cos x} - \cancel{\cos x} - \cos^2 x}$$

$$\frac{\sin^2 x + \sin^2 x}{1 - \cos^2 x} = \frac{2\cancel{\sin^2 x}}{\cancel{\sin^2 x}} = 2$$

$$\therefore \quad \frac{\sin^2 x}{1 - \cos^2 x} + \frac{\sin^2 x}{1 + \cos x} = 2$$

4. 3 COMPLEMENTERY ANGLES

Two angles are said to be complementary if their sum is 90^0



Angle A and angle C are complementary

$\hat{A} + \hat{C} = 90^0$

$\therefore$ If $\hat{C} = \theta$, then $\hat{A} = 90 - \theta$.

From the figure above

Sin θ =cos (90 - θ)

Cos θ = sin (90 - θ)

$$\text{Tan } \theta = \frac{1}{\text{Tan } (90 - \theta)} = \cot (90 - \theta)$$

$$\therefore \tan (90 - \theta) = \frac{1}{\tan \theta} = \cot \theta$$

Examples 4. 5

Solve the equation

(a) $\sin \theta = \cos 40^0$

(c) $\cos \theta = \sin (\theta + 22^0)$

(d) $\sin 3\theta = \cos 2\theta$

Solution

(a) $\sin \theta = \cos 40^0$

but $\sin\theta = \cos (90 - \theta)$

$\therefore \cos (90 - \theta) = \cos 40^0$

$90 - \theta = 40^0$

$\theta = 90^0 - 40^0$

$\theta = 90^0$

(b) $\cos \theta = \sin (\theta + 22^0)$ [$\cos \theta = \sin (90 - \theta)$]

$90 - \theta = \theta + 22^0$

$90^0 - 22^0 = \theta + \theta$

$68^0 = 2\theta$

$\frac{68}{2} = \theta$

$34^0 = \theta$

(c) $\sin 3\theta = \cos 2\theta$ ($\sin 3\theta = \cos (90 - 3\theta)$)

$\therefore \cos(90 - 3\theta) = \cos 2\theta$

$90 = 2\theta + 3\theta = 5\theta$

$\frac{90}{5} = \theta$

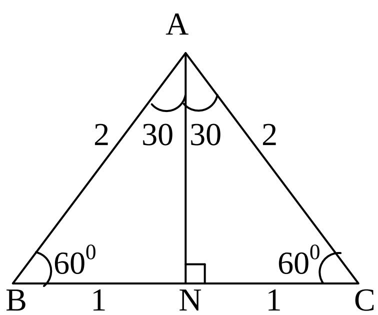
$\theta = 18^0$

4.4 TRIGONOMETRIC RATIO OF SPECIAL ACUTE ANGLES

These are angles whose values can be determined without the use of mathematical tables or calculator

Examples are 0^0, 30^0, 45^0, 60^0 and 90^0.

(i) 30^0 and 60^0 - these can be produced from an equilateral triangle of 2 units sides each.



Using Pythagoras theorem, $AN^2 = AC^2 - NC^2$

$= 2^2 - 1^2 = 4 - 1$

$AN = \sqrt{3}$

Using any of the 30^0 in the figure above

$\sin 30^0 = \frac{1}{2}$

$\cos 30^0 = \frac{\sqrt{3}}{2}$

Tan 30^0 = $\frac{1}{\sqrt{3}}$

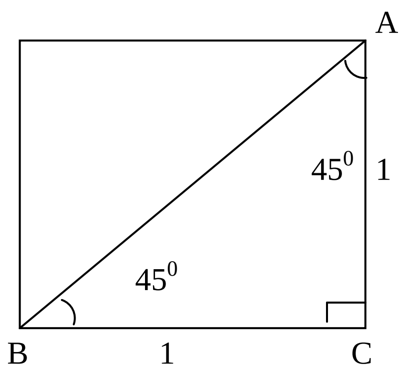
Using any of the 60^0 in the figure above

Sin 60^0 = $\frac{\sqrt{3}}{2}$

Cos 60^0 = $\frac{1}{2}$

Tan 60^0 = $\frac{\sqrt{3}}{1}$.

(ii) 45^0 – this can be produced from a square of unit sides with in of the diagonal



Using Pythagoras theorem, $AB^2 = 1^2 + 1^2$

$AB^2 = 1 + 1$

$AB = \sqrt{2}$

$\therefore$ sin 45^0 = $\frac{1}{\sqrt{2}}$

cos 45^0 = $\frac{1}{\sqrt{2}}$

tan 45^0 = $\frac{1}{1}$ = 1

(iii) 0^0 and 90^0

sin 0 = 0

cos 0 = 1

$$\tan 0 = \frac{\sin 0}{\cos 0} = \frac{0}{1} = 0$$

$\sin 90^0 = 1$

$\cos 90^0 = 0$

$$\tan 90^0 = \frac{\sin 90^0}{\cos 90^0} = \frac{1}{0} = \infty \text{ (infinity)}$$

Example 4.6

If $\sin\theta = \frac{\sqrt{3}}{2}$ and θ is less than 90^0,

Calculate $\frac{\tan(90 - \theta)}{\cos^2\theta}$.

Solution

$$\frac{\tan(90 - \theta)}{\cos^2\theta} = \frac{\cot\theta}{\cos^2\theta}$$

$$= \frac{\frac{\cos\theta}{\sin\theta}}{\cos^2\theta} = \frac{\cos\theta}{\sin\theta} \times \frac{1}{\cos^2\theta}$$

$$= \frac{1}{\text{Sin}\theta} \times \frac{1}{\cos\theta} = \frac{1}{\text{Sin}\theta \cos\theta}$$

But $\cos^2\theta = 1 - \sin^2\theta$

$$\text{Cos}\,\theta = \sqrt{1 - \sin^2\theta}$$

$$= \sqrt{1 - \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \sqrt{1 - \frac{3}{4}}$$

$$= \sqrt{\frac{1}{4}} = ½$$

$$\therefore \frac{\tan(90 - \theta)}{\cos^2\theta} = \frac{1}{\sin\theta\cos\theta} = \frac{1}{\frac{\sqrt{3}}{2} \times \frac{1}{2}}$$

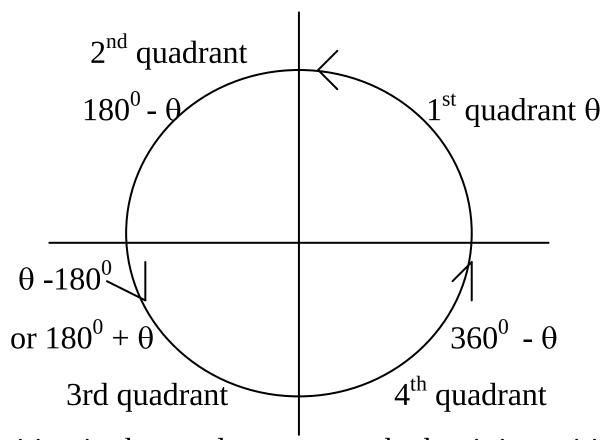
$$= \frac{1}{\frac{\sqrt{3}}{4}} = \frac{4}{\sqrt{3}}$$

4.5 TRIGONOMETERY OF OBTUSE, REFLEX, ALLIED AND NEGATIVE ANGLE.

An obtuse angle is the angle greater than 90^0 but less than 180^0. A reflex angle is the greater that 180^0 but less than 360^0 while an allied angle is the angle greater than 360^0.

Trigonometrical ratios of angle other than acute are found by expressing them in a form that contains an acute angle. The corresponding acute angle is obtained by illustrating the given angle as a displacement from a fixed point or line in anti clockwise direction.

Generally, all angles are measured from the positive side of x – axis on x –y plane and the required acute angle is the one made with the y- axis. The combination of any x –y plane is called quadrant and since measurement is made from the positive axis of x, the quadrants are numbered in the same way as shown below



The position in the quadrant states whether it is positive or negative

S (students) Sin = + ve	A (All) All are +ve
T(Try) Tan = + ve	C (commerce) Cos + ve

(All students try commerce) or CAST

EQUATION

5.1 LINEAR EQUATION

A linear equation in a single variable (unknown) involves powers of the variable no higher than the first. It is also referred to as a simple equation.

Example 5.1

Solve 3 (2 – x) = 14 + 5x

Solution

$3(2 - x) = 14 + 5x$

$6 - 3x = 14 + 5x$

$-3x - 5x = 14 - 6$

$-8x = 8$

$x = \frac{-8}{8} = -1$

Example 5.2

5 (x - 1) + 3 (2x + 9) – 2 = 4 (3x – 1) + 2 (4x + 3)

Solution

$5(x - 1) + 3(2x + 9) - 2 = 4(3x - 1) + 2(4x + 3)$

$5x - 5 + 6x + 27 - 2 = 12x - 4 + 8x + 6$

$5x + 6x + 27 - 2 - 5 = 12x + 8x + 6 - 4$

$11x – 20 = 20x + 2$

$11x – 20x = 2- 20$

$- 9x = - 18$

$x = \frac{18}{9} = 2$

Example5.3

Three boys shared some oranges. The first received $\frac{1}{3}$ of the oranges. The second received $\frac{2}{3}$ of the remainder, if the third boy received the remaining 12 oranges. How many oranges did they share?

Solution

Let the total oranges = x

1^{st} boy received $\frac{1}{3}x$

Remainder $= x - \frac{1}{3}x = 2/3x$

2^{nd} boy received $\frac{2}{3}$ of the remainder $= \frac{2}{3}(2/3x)$

$= \frac{4}{9}x$

If the 2nd boy received $\frac{2}{3}$ of the remainder, the 3^{rd} boy will receive $\frac{1}{3}$ of the remainder

$\therefore$ 3^{rd} boy received $\frac{1}{3}(\frac{2}{3}x) = \frac{2}{9}x$

But $\frac{2}{9}x = 12$

6

$$x = \frac{9 \times \cancel{12}}{\cancel{2}_{1}} = 54$$

$\therefore$ the number of oranges shared = 54

Example 5.4

A man spent $\frac{1}{10}$ of his monthly salary on rent and 5/6 of the remainder on household needs . If ₦1800.00 is left, how much is his monthly salary.

Solution

Let the man's monthly salary = x

Rent $\Rightarrow \frac{1}{10}x$

Remainder $\Rightarrow \frac{x}{1} - \frac{1}{10}x = \frac{10x - x}{10} = \frac{9x}{10}$

Household needs = $\frac{5}{6}$ of remainder

$$= \frac{\cancel{5}}{\cancel{6}_{2}} \times \frac{\cancel{9x}^{3}}{\cancel{10}_{2}} = \frac{3x}{4}$$

fraction of remainder left = $(1 - \frac{5}{6}) \times \frac{9x}{10}$

$$= \frac{1}{\cancel{6}_{2}} \times \frac{\cancel{9}^{3}x}{10} = \frac{3x}{20}$$

$\therefore \frac{3x}{20} = 1800$

$3x = 1800 \times 20$

$$x = \frac{\overset{600}{\cancel{1800}} \times 20}{\cancel{3}} = ₦12,000.00$$

The man's monthly salary = ₦12, 000.00

5.2 SIMULTANEOUS EQUATIONS

Example 5.5

Solve the pair of equations

$5x + 2y = 14$

$3x - 4y = 24$

SOLUTION BY SUBSTITUTION

$5x + 2y = 14$ --------(i)

$3x - 4y = 24$ ---------(ii)

from equation (i)

$5x + 2y = 14$

$5x = 14 – 2y$

$x = \frac{14 - 2y}{5}$ -----------(iii)

put $x = \frac{14 - 2y}{5}$ in equation(ii)

$3(\frac{14 - 2y}{5}) - 4y = 24$

$- 4y = 24$

multiplying through by 5,

$42 – 6y – 20y = 120$

$42 - 26y = 120$

$-26y = 120 – 42$

$-26y = 78$

$$y = \frac{-78}{26} = -3$$

put $y = -3$ in equation (iii)

$$x = \frac{14 - 2y}{5} = \frac{14 - 2(-3)}{5} = \frac{14 + 6}{5} = \frac{20}{5} = 4$$

$x = 4$, $y = -3$

Example 5.6

Solve $3x + 2y = 16$ and $4x - 3y = 10$

SOLUTION BY EQUATING COEFFICIENTS

$3x + 2y = 16$ --------(i)

$4x – 3y = 10$ ---------(ii)

multiply equation (i) by 4 (the coefficient of x in equation(ii)) and equation (ii) by 3 (the coefficient of x in equation (i))

$4(3x + 2y = 16) \Rightarrow 12x + 8y = 64$ ---------(iii

$3(4x - 3x = 10) \Rightarrow 12x - 9y = 30$ -----------(iv)

subtract equation (iv) from(iii)

$17y = 34$

$$y = \frac{34}{17} = 2$$

put $y = 2$ in equation (i)

$3x + 2y = 16$

$3x + 2(2) = 16$

$3x + 4 = 16$

$3x = 16 - 4 = 12$

$$x = \frac{12}{3} = 4$$

$$x = 4, \ y = 2$$

Example 5.7

Solve the simultaneous equations

$$\frac{2}{x} - \frac{3}{y} = 2$$

$$\frac{4}{x} + \frac{3}{y} = 10$$

Solution

$$\frac{2}{x} - \frac{3}{y} = 2 \Rightarrow 2(\frac{1}{x}) - 3(\frac{1}{y}) = 2$$

$$\frac{4}{x} + \frac{3}{y} = 10 \Rightarrow 4(\frac{1}{x}) + 3(\frac{1}{y}) = 10$$

Let $\frac{1}{x} = A$ and $\frac{1}{y} = B$

$\therefore$ 2A - 3B = 2 -------(i)

4 A+ 3B = 10-----(ii)

Adding equation (i) and (ii),

6A = 12

$$A = \frac{12}{6} = 2$$

Put A = 2 in equation (i)

2A - 3B = 2

2(2) - 3B = 2

4 – 3B = 2

-3B = 2 - 4

= 2

$B = 2/3$

But $\frac{1}{x} = A$ and $\frac{1}{y} = B$

$\therefore \frac{1}{x} = 2$ and $\frac{1}{y} = \frac{2}{3}$

$x = \frac{1}{2}$ and $2y = 3$

$y = 3/2$

$\therefore x = ½, y = 3/2$

Example 5.8

A car painter charges ₦40 per day for himself and ₦ 10 per day for his assistant. If a fleet of cars were painted for ₦2,000 and the painter worked 10 days more than his assistant, how much did the assistant receive

SOLUTION

Let the painter work for x days and the assistant for y days

The amount received by painter $= 40x$

The amount received by assistant $= 10y$

Total amount received $= 40x + 10y = 2000$

$\therefore 40x + 10y = 2000$ -------(i)

Also $x – y = 10$ --------(ii)

From equation (ii) $x – y = 10$

$x = 10 + y.$

Put $x = 10 + y$ in equation (i)

$40(10 + y) + 10y = 2000$

$400 + 40y + 10y = 2000$

$400 + 50y = 2000$

$50y = 2000 – 400 = 1600$

$y = \frac{1600}{50} = 32$

Amount received by assistant = 10y

$= 10 \times 32 =$ ₦ 320

89

Examples 5.9

Solve $5x - 3y - 2z = 31$

$2x + 6y + 3z = 4$

$4x + 2y - z = 30$

Solution

$5x - 3y - 2z = 31$

$2x + 6y + 3z = 4$

$4x + 2y - z = 30$

Take a pair of equations and eliminate one of the variables

$5x - 3y - 2z = 31$ -----(i)

$2x + 6y + 3z = 4$ ------(ii)

multiply equation (i) by 3 $\Rightarrow$ $15x - 9y - 6z = 93$ - (iv)

multiply equation (ii) by 2 $\Rightarrow$ $4x + 12y + 6z = 8$ ----(v)

adding equation(iv) and (v) $19x + 3y = 101$ -------(vi)

$5x - 3y - 22 = 31$ -----------(i)

$4x + 2y - 2 = 30$ --------------(ii)

multiply equation (iv) by 2 and equation (i) by 1

$5x - 3y - 2z = 31$

$-$ $\underline{8x + 4y - 2z = 60}$

$- 3x - 7y = - 29$

$-(3x + 7y) = - 29$

$3x + 7y = 29$ -----------(vii)

90

Bring equations vi and vii together,

$19x + 3y = 101$ --------(vi)

$3x + 7y = 29$ ----------(vii)

multiply equation (vi)by 7 and equation (vii) by 3

$$133x + 21y = 707$$

$$\underline{- \quad 9x + 21y = 87}$$

$$124x = 620$$

$$x = \frac{620}{124} = 5$$

put x = 5 in equation (vii)

$$3x + 7y = 29$$

$$3(5) + 7y = 29$$

$$15 + 7y = 29$$

$$7y = 29 - 15 = 14$$

$$y = \frac{14}{7} = 2$$

put x = 5 and y = 2 in equation (ii)

$$4x + 2y - z = 30$$

$$4(5) + 2(2) - z = 30$$

$$20 + 4 - z = 30$$

$$24 - z = 30$$

$$- z = 30 - 24 = 6$$

$$z = - 6$$

$$\therefore x = 5, y = 2, z = -6$$

5.3 QUADRATIC EQUATION

Quadratic equations can be solve by

(i) factorisation

(ii) completing the square

(iii) formular method

(iv) graphical method,

A quadratic equation involves no powers of the variables involved greater than the second and takes the general form

$$y = ax^2 + bx + c$$

Solution by factorisation

Example 5.10

Solve $x^2 - 9x + 18 = 0$

Solution

$x^2 - 9x + 18 = 0$

$(x - 3)(x - 6) = 0$

$x - 3 = 0$ or $x - 6 = 0$

$x = 3$ or 6

Example 5.11

solve $3x^2 + 14x + 8 = 0$

Solution

Solve $3x^2 + 14x + 8 = 0$

multiply $3x^2$ by 8 $\Rightarrow (24x^2)$

then look for factors whose product will give $24x^2$ and whose sum will give $14x$

then two factors are $12x$ and $2x$.

$\therefore\ 3x^2 + 14x + 8 = 0$

$(3x^2 + 12x) + (2x + 8) = 0$

$3x(x + 4) + 2(x + 4) = 0$

$(3x + 2)(x + 4) = 0$

$\therefore$ 3x + 2 = 0 or x + 4 = 0

3x = - 2 or - 4

x = - 2/3

$\therefore$ x = -2 /3 or -4

Example 5.12

Solve $2x^2 - 3x - 5 = 0$

Solution

Two factors whose product will give $-10x^2$ and whose sum will give – 3x are -5x and + 2x .

$\therefore$ $2x^2 - 3x - 5 = 0$

$(2x^2 + 2x) (-5x - 5) = 0$

2x (x + 1) -5 (x + 1) = 0

(2x – 5) (x + 1) = 0

$\therefore$ 2x – 5 = 0 or x + 1 = 0

2x = 5 or – 1

x = 5/2

x = -1 or 5/2

solution by completing the square

Example 5.13

Solve $3x^2 + 8x + 4 = 0$

Solution

$3x^2 + 8x + 4$

Divide through by 3 (the coefficient of x^2)

$x^2 + \frac{8}{3}x = \frac{-4}{3}$

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find $\frac{1}{2}$ of the coefficient of x and add the square to both sides of the equation

$$x^2 + \frac{8}{3}x + (\frac{4}{3})^2 = \frac{-4}{3} + (\frac{4}{3})^2 \quad \left(\frac{1}{2} \times \frac{8}{3} = \frac{4}{3}\right)$$

$$(x + \frac{4}{3})^2 = \frac{-4}{3} + \frac{16}{9} = \frac{-12+16}{9} = \frac{4}{9}$$

$$(x + \frac{4}{3})^2 = \frac{4}{9}$$

$$x + \frac{4}{3} = \pm\sqrt{\frac{4}{9}} = \pm\frac{2}{3}$$

$$x = -\frac{4}{3} \pm \frac{2}{3} = \frac{-4 \pm 2}{3}$$

$$\therefore x = \frac{-4+2}{3} \text{ or } \frac{-4-2}{3}$$

$$x = \frac{-2}{3} \text{ or } \frac{-6}{3}$$

$$x = \frac{-2}{3} \text{ or } -2$$

Example 5.14

Solve $x^2 + 4x - 6 = 0$

Solution

$x^2 + 4x - 6 = 0$

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$x^2 + 4x = 6$

Add the square of $\frac{1}{2}$ to the coefficient of x to both sides of equation

$x^2 + 4x + 2^2 = 6 + 2^2 = 6 + 4 = 10$

$(x + 2)^2 = 10$

$x + 2 = \pm\sqrt{10}$

$x = -2 \pm \sqrt{10}$

$x = -2 + \sqrt{10}$

or $-2 - \sqrt{10}$

Solution by formula method

Given that $ax^2 + 6x + c = 0$

Then $x = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

Example 5.15

Solve $2x^2 - 3x – 5 = 0$

Solution

$2x^2 - 3x - 5 = 0$

here $a = 2, b = -3, c = -5$

$x = \dfrac{b \pm \sqrt{b^2 - 4ac}}{2a}$

$x = \dfrac{-(-3) \pm \sqrt{(-3)^2 - 4x\ 2x - 5}}{2 \times 2}$

$x = \dfrac{3 \pm \sqrt{9 + 4\ 0}}{4} = \dfrac{3\pm \sqrt{49}}{4}$

$x = \dfrac{3\pm 7}{4}$

$x = \dfrac{3 + 7}{4}$ or $\dfrac{3 - 7}{4}$

$x = \dfrac{10}{4}$ or $\dfrac{-4}{4} = 5/2$ or -1

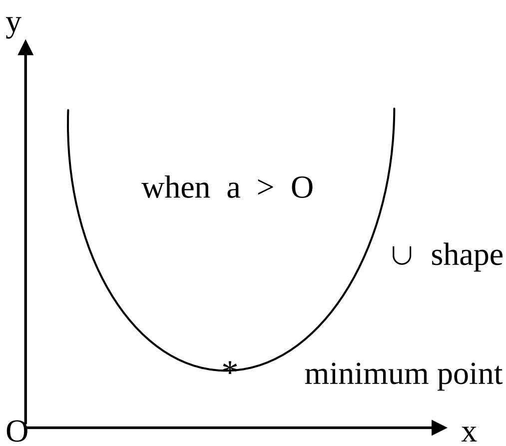
x = -1 or 5/ 2

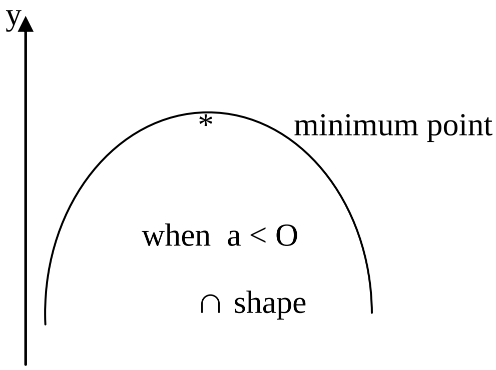
Solution by Graphical method

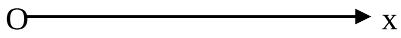
The graph of a quadratic function is called a parabola and takes on a distinctive shape

(i) ∪ shape if a > O, which includes a distinct local minimum point where the curve turns

(ii) ∩ shape if a < O which includes a distinct local maximum point where the curve tune







Example 5.16

Plot the graph of the function

$y = x^2 - 4.5x + 3.5$ between $x = 0$ and $x = 4$

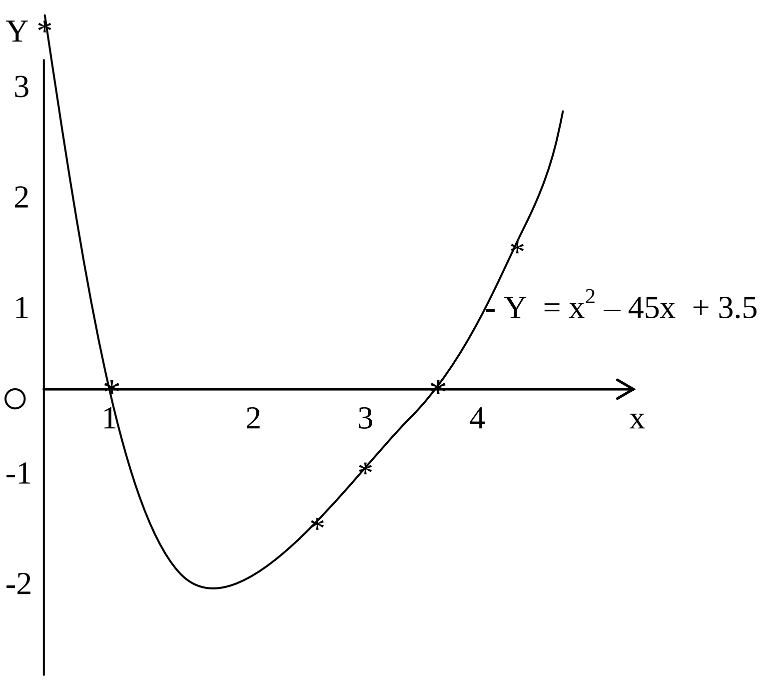
and find the solution of the equation

$x^2 - 4.5x + 3.5 = 0$ from the graph

SOLUTION

X	0	1	2	3	4
X^2	0	1	4	9	16
- 4. 5x	0	- 4 .5	- 9	- 13 .5	- 18
3. 5	3.5	3.5	3. 5	3. 5	3. 5
Y	3. 5	0	-1 .5	- 1 .0	1.5

The value of Y are plotted against X as shown below



The solution of quadratic equation $x^2 - 4.5x + 3.5 = 0$ is defined by points on x –axis where the curve crosses it x = 1 or 3.5

Example 5.17

A Company invests in a particular project and it has been estimated that after X months of running, the cumulative profit (₦000) from the project is given by the function $31.5x - 3x^2 - 60$, where x represents time in month. The project can run for nine months at the most.

(a) Draw a graph which represents the profit function

(b) Calculate the break even time points for the project

(c) What is the initial cost of the project

(d) Use the graph to estimate the best time to end the project

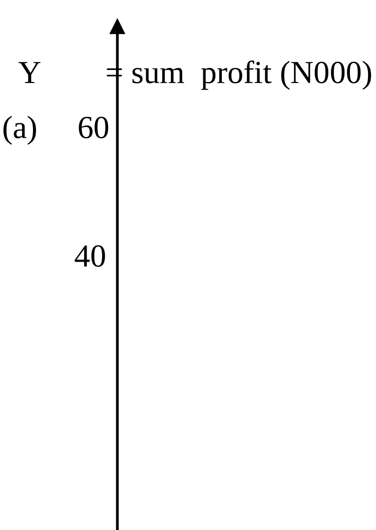
Solution

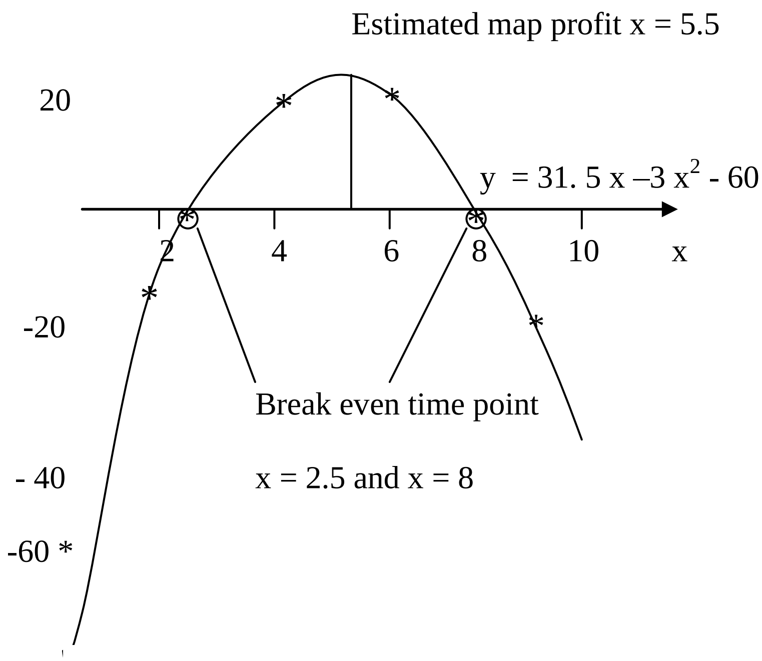
Profit $y = 31.5x - 3x^2 - 60$,

The lowest relevant value of x = 0 and the highest is x = 9 since nine months is the maximum length of the project

X	0	2	4	6	8	9
31 .5x	0	63	126	189	252	283.5
- 3x2	0	- 12	- 48	- 108	- 192	- 243
- 60	- 60	- 60	- 60	- 60	- 60	- 60
Y	- 60	- 9	18	21	0	- 19.5

(a)





Estimated map profit x = 5.5
20
y = 31. 5 x –3 x2 - 60
2
4
6
8
10
x
-20
Break even time point
- 40
x = 2.5 and x = 8
-60 *

SEQUENCE

A sequence or progression is a succession of terms in such a way that the terms are related to one another according to a well defined rule. The nth term of a general sequence is denoted by U_n.

Examples are given below

(i) 11, 17, 23, 29, - - - -

(ii) 1, 4, 9, 16 - - -

(iii) 1, 2, 4, 8, - - -

(iv) – 6, - 3 , $-1^{1/2}$, - ¾, - - -

In (i) each terms is 6 more than the preceding term

In (ii) , the sequence is the set of square numbers 1^2, 2^2, 3^2, - - -

In (iii) each term is double the preceding term and

In (iv) each term is half the preceding term.

7.1 SEQUENCE AS A FUNCTION

Example 7.1

Consider the sequence 2, 5, 8, 11 - - -

The table below shows how each term is related to the positive integers 1, 2, 3, 4, - - - - , n

Number of terms	1^{st} n = 1	2^{nd} n =2	3^{rd} n = 3	4^{th} n =4	nth
Term	2 2+0x3	2+3 2+1x3	2+3+3 2+2x3	2+3+3+3 2+3x3	2+(n – 1) x3

∴ The nth term U_n = 2 + (n – 1) x3

= 2 + 3n – 3

= U_n = 3n - 1

This gives us a formula for finding any required term.

In general, a sequence U_1, U_2, U_3, - - -U_n, is the set of images given by the function U_n = f(n) of the positive integers 1, 2, 3, - - n.

Example 7.2

Given the sequence 5, 1, -3, –7, - - - . Find a formula for the nth term.

Solution

U_1 = 5 = 5 – 0 x 4

U_2 = 5 – 4 = 5 – 1 x4

U_3 = 5 – 4 - 4 = 5 – 2x 4

U_4 = 5 – 4 – 4 – 4 = 5 – 3 x 4

U_n = 5 – (n –1) x 4 = 5 – 4n + 4 = 9 – 4n

U_n = 9 – 4n

Example 7.3

If $U_n = (-1)^n$ x n , find U_1, U_2, and U_3

Solution

$U_n = (-1)^n$ x n

$U_1 = (-1)^1$ x 1 = - 1

U_2 = 1 x 2 = 2

$U_3 = (-1)^3$ x 3

U_3 = -1 x 3 = -3

Example 7.4

The nth term of a sequence is given by

$U_n = 2 + 3U_{n-1}$ while $U_4 = 36 + U_3$, find U_2.

Solution

Given , $U_n = 2 + 3U_{n-1}$, $U_4 = 36 + U_3$ ----------(i)

If $n = 4$, $U_4 = 2 + 3U_{4-1}$

$U_4 = 2 + 3U_3$ ------------------(ii)

Equating (i) and (ii)

$36 + U_3 = 2 + 3U_3$

$36 - 2 = 3U_3 - U_3$

$34 = 2U_3$

$\frac{34}{2} = U_3$

$17 = U_3$

if $n = 3$, then $U_3 = 2 + 3U_{3-1}$

$U_3 = 2 + 3U_2$ $(U_3 = 17)$

$17 = 2 + 3U_2$

$17 - 2 = 3U_2$

$15 = 3U_2$

$U_2 = \frac{15}{3}$

$\therefore U_2 = 5$

7.2 SERIES

A series is the addition of the terms of a sequence. The sum of the nth terms of the sequence U_1, U_2, U_3, - - - U_n is generally denoted by S_n.

Example 7.5

1. Find the series of the 1st four terms of the sequence

$U_n = 2^n + 4n^2$

Solution

$U_n = 2^n + 4n^2$

$U_1 = 2^1 + 4(1)^2 = 2 + 4 = 6$

$U_2 = 2^2 + 4(2)^2 = 4 + 16 = 20$

$U_3 = 2^3 + 4(3)^2 = 8 + 36 = 44$

$U_4 = 2^4 + 4(4)^2 = 16 + 64 = 80$

Series = 6 + 20 + 44 + 80 = 150

Example 7.6

Find the series of the first three terms of the sequence

$U_n = n^2 + 3n - 2$

Solution

$U_n = n^2 + 3n - 2$

$U_1 = 1^2 + 3(1) - 2 \quad = 1 + 3 - 2 = 2$

$U_2 = 2^2 + 3(2) - 2 \quad = 4 + 6 - 2 = 8$

$U_3 = 3^2 + 3(3) - 2 \quad = 9 + 9 - 2 = 16$

Series $= U_1 + U_2 + U_3 = 2 + 8 + 16 = 26$

7.3 ARITHMETIC PROGRESSION

Arithmetic Progression (A.P) is a type of sequence in which the consecutive terms of the sequence differ by a constant number.

Generally, arithmetic progression is represented as below

a, a+d, a +2d, a +3d, - - - , a + (n – 1) d. where a = first term

d = common difference $= U_2 - U_1 = U_3 - U_2 = U_n - U_{n-1}$

n = number of terms.

The following should be noted for an arithmetic progression

(i) The nth term, $U_n = a + (n - 1)d$

(ii) The last term, $l = a + (n - 1) d$

(iii) Common difference,

$d = U_n - U_{n-1} = U_2 - U_1 = U_3 - U_2$

(iv) Sum of n terms, $S_n = \frac{n}{2}(2a + (n-1)d)$

$$\text{or } S_n = \frac{n}{2}(a + l)$$

Example 7.7

Find the 10^{th} terms of the sequence –23, – 17, – 11, - - -

Solution

– 23, – 17 , –11, - - -

$d = U_2 - U_1 = -17 - (-23) = -17 + 23 = 6$, $a = -23$

nth term, $U_n = a + (n-1)d$

10^{th} term, $U_{10} = a + (10-1)d = a + 9d$

$= -23 + 9d = -23 + 54$

$= 31$

Example 7.8

The first three terms of an A.P. is given as x + 1, 3x –18 and 2x – 1. Find x and the 8^{th} term.

Solution

Given x + 1, 3x –18 and 2x – 1

For an A. P, common difference, $d = U_2 - U_1 = U_3 - U_2$

$\therefore 3x - 18 - (x + 1) = 2x - 1 - (3x - 18)$

$3x - 18 - x - 1 = 2x - 1 - 3x + 18$

$2x - 19 = -x + 17$

$2x + x = 17 + 19$

$3x = 36$

$x = 36/3 = 12$

The sequence becomes 13, 18, 23, - - -

$\therefore$ a = 13, d = 18 – 13 = 5

8^{th} term, U_8 = a + 7d

= 13 + 7 (5) = 13 + 35 = 48

Example 7.9

The 4^{th} term of an AP is 13 while the 10^{th} term is 31. Find the 21^{st} term.

Solution

4^{th} term U_4 = a + 3d

a + 3a = 13 -----------------(i)

10^{th} term, U_{10} + 9d

a + 9d = 31 -------------(ii)

subtracting equation (i) from (ii)

6d = 18

d = 18/6 = 3

substituting d = 3 in equation (i)

a + 3d = 13

a + 3(3) = 13

a + 9 = 13

a = 13 – 9 = 4

$\therefore$ 21^{st} term U_{21} = a + 20 d

= 4 + 20 (3)

= 4 + 60 = 64

Example 7.10

A man is able to save N50 of his salary in a particular year. After this, every year he saved N20 more than the preceding year . How long does it take him to save N4, 370.

Solution

The problem is an A. P with

a = N50, d = N20, S_n = N4370

Sum of n terms, $S_n = \frac{n}{2}(20+(n-1)d)$

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$4370 = \frac{n}{2}(2(50) + (n-1)20)$

$4370 = \frac{n}{2}(100 + 20n - 20)$

$2 \times 4\,370 = n(80 + 20n)$

$8740 = 80n + 20n^2$

$\therefore 20n^2 + 80n - 8740 = 0$

Dividing through by 20, we have

$n^2 + 4n - 437 = 0$

$(n - 19)(n + 23) = 0$

$n - 19 = 0$ or $n + 23 = 0$

$n = 19$ or -23

But n cannot be negative

$\therefore n = 19$

It is going to take 19 years to save N4370.

Example 7.1

Find the sum of the first twenty terms of the arithmetic progression log a, $\log a^2$, $\log a^3$, - - - -

Solution

Log a, $\log a^2$, $\log a^3$, - - - -

$a = \log a$, $d = \log a^2 - \log a = \log \frac{a^2}{a} = \log a$

$n = 20$

$S_n = \frac{n}{2}\ (2a + (n-1)\ d)$

$S_{20} = \frac{20}{2}\ (2\log a + (20 - 1)\log a)$

$= 10\ (2\log a + 20\log a - \log a)$

$= 10\ (22\log a - \log a)$

$= 10 \times 21\log a$

$= 210\log a = \log a^{210}$

7.4 GEOMETRIC PROGRESSION (G. P)

This is a sequence with a common ratio between the consecutive terms. Generally a geometric progression is represented as

$a, ar, ar^2, ar^3, \ - - - , ar^{n-1}$

where a = first term

r = common ratio

n = number of terms.

The following should be noted for a geometric progression

(i) Common ratio, $r = \frac{U_n}{U_{n-1}} = \frac{U_2}{U_1} = \frac{U_3}{U_2}$

(ii) The nth term , $U_n = ar^{n-1}$

(iii)The last term, $l = ar^{n-1}$

(iv) Sum of n terms, $S_n = \frac{a(1 - r^n)}{1 - r}$ where $r < 1$

or $S_n = \frac{a(r^n - 1)}{r - 1}$ where $r > 1$

(v) Sum to infinity, $S_\infty = \frac{a}{1 - r}$ where $|r| < 1$

Example 7.12

1. Given the progression 27, 9, 3, - - - . Find the 5th and nth term.

Solution

27, 9, 3, - - -

a =27, r $= \frac{9}{27} = \frac{1}{3}$

5^{th} term, $U_5 = ar^{5-1} = a\,r^4$

$$U_5 = 27\left(\frac{1}{3}\right)^4 = \overset{1}{\cancel{27}} \times \frac{1}{\underset{3}{\cancel{81}}} = \frac{1}{3}$$

$$U_5 = \frac{1}{3}$$

nth term , $= U_n = ar^{\,n-1}$.

$$U_n = 27\left(\frac{1}{3}\right)^{n-1}$$

$$= 3^3 \times \frac{1}{3^{n-1}} = \frac{3^3}{3^{n-1}} = 3^{3-(n-1)}$$

$$U_n = 3^{3-n+1} = 3^{4-n}$$

Example 7.13

If 5, x, y, 40 are in geometrical progression, find x and y respectively

Solution

5, x, y, 40

a = 5, $U_4 = 40$

4^{th} term, $U_4 = ar^3$

$$\therefore\ ar^3 = 40$$

$$5r^3 = 40$$

$$r^3 = \frac{40}{5} = 8$$

$$r = \sqrt[3]{8} = 2$$

x = 2nd term, $U_2 = ar$

$$= 5 \times 2 = 10$$

$y = 3rd\ term,\ U_3 = ar^2$

$= 5 \times 2^2 = 5 \times 4 = 20$

Example 7.14

x + 1, 2x – 1 and 3x + 1 are three consecutive terms of a geometric progression . Find the possible values of the common ratio.

Solution

x + 1, 2x – 1 and 3x + 1

Common ratio, $r = \frac{U_2}{U_1} = \frac{2x - 1}{x + 1}$

$r = \frac{U_3}{U_2} = \frac{3x + 1}{2x - 1}$

$\therefore \frac{2x - 1}{x + 1} = \frac{3x + 1}{2x - 1}$

cross multiply

$(2x - 1)(2x - 1) = (3x + 1)(x + 1)$

$4x^2 - 2x - 2x + 1 = 3x^2 + 3x + x + 1$

$4x^2 - 4x + 1 = 3x^2 + 4x + 1$

$4x^2 - 3x^2 - 4x - 4x = 0$

$x^2 - 8x = 0 \Rightarrow x(x - 8) = 0$

$x = 0$ or $x - 8 = 0$

$\therefore x = 0$ or 8

when x = 0

$r = \frac{2x - 1}{x + 1} = \frac{2(0) - 1}{0 + 1} = \frac{-1}{1} = -1$

when n = 8

$r = \frac{2x - 1}{x + 1} = \frac{2(8) - 1}{8 + 1} = \frac{16 - 1}{9} = \frac{15}{9} = \frac{3}{2}$

$\therefore r = -1$ or $5/3$