CONDITIONAL PROBABILITY

Most work we have considered so far involved independent events. As a result, getting the probability of such an event was reasonably straightforward. However, when the events are dependent, then solving them become more complicated.

Conditional Probability written as P(B/A) is the probability of an event B given that a "previous" event A has occurred. The conditional probability of B given A is

$$P(B/A) = \frac{P(\text{both A and B})}{P(A)} = \frac{P(A \cap B)}{P(A)} \qquad (4.8)$$

If two events are independent events then

$$P(B/A) = P(B) \quad \text{or} \quad P(A/B) = P(A) \qquad (4.9)$$

Example 4.19: A coin is tossed thrice. Find the probability that there are two heads (i) given that at least one is a tail (ii) given that the first is a tail.

Solution

S = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

i) Sample space for at least one tail

S = {HHT, HTH, THH, HTT, THT, TTH, TTT} having 7 elements. n(2 heads) = 3

Therefore, the conditional probability that there are 2 heads, given that at least one is a tail, is

P(2 heads/at least one is a tail) = 3/7

ii. Sample space for the first being tail
= {THH, THT, TTH, TTT} having 4 elements. n(2 heads) = 1.

Therefore, the conditional probability that there are two heads given that the first is a tail is

P(2 heads/first is a tail) = ¼

USING THE FORMULA

i. Let A: at least one is a tail
B: 2 heads appear

The conditional probability is

P(B/A) = P(2 heads appear/at least one is a tail)

$$= \frac{P(\text{2 heads appear and at least one is a tail})}{P(\text{at least one is a tail})}$$

Using the original sample space of all 8 equally likely possible outcomes, we see that

P(at least one is a tail) = 7/8 and
P(2 heads appear and at least one is a tail) = 3/8

Therefore,

$$P(B/A) = P(\text{2 heads appears/at least one is a tail}) = \frac{3/8}{7/8}$$

$$= 3/8 \times 8/7 = 3/7$$

which is the same result as we obtained above.

ii. P(the first is a tail) = 4/8

P(2 heads appear and the first is a tail) = 1/8

Therefore,

P(2 heads appear/the first is a tail)

$$= \frac{P(\text{2 heads appear and the first is a tail})}{P(\text{the first is a tail})}$$

$$= \frac{1/8}{4/8}$$

$$= 1/8 \times 8/4 = 1/4$$

Example 4.20: A die to tossed twice. Find (i) probability of getting a sum of 9 (ii) probability of getting a sum of 9 giving that the number on the 2nd toss is larger than the number on the first toss.

Solution

+	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

i. P(getting a sum of 9) $= \frac{4}{36} = \frac{1}{9}$

iii. Let A = "sum of 9", B = "2^{nd} toss number larger than the first toss".

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

P(B) = P(2^{nd} toss number larger than the first toss) $= \frac{15}{36}$

P(A ∩ B) = P(sum of 9 and 2^{nd} toss number larger than the first toss) $= \frac{2}{36}$

Therefore,

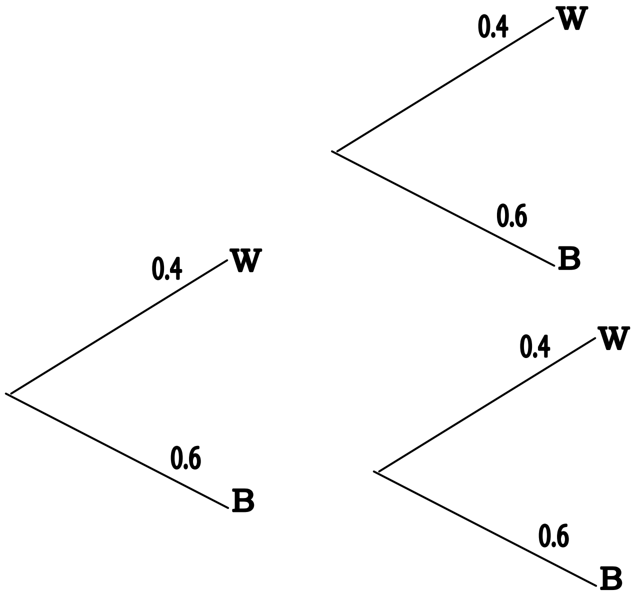
$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

$$= \frac{2/36}{15/36} = 2/15$$

Example 4.21: A bag contains 10 balls. Four are white and 6 are black. Draw the tree diagram when two balls are drawn with (i) replacement (ii) no replacement

Solution

(i) with replacement   Figure 4.3 Tree Diagram



P(2^{nd} white) = P(2^{nd} is white and 1^{st} is white) or

P(2^{nd} is white and 1^{st} is black)

= P(1^{st} is white) – P(2^{nd} is white given the 1^{st} is white)
+ P(1^{st} is black). P(2^{nd} is white given the 1^{st} is black)

= 0.4 x 0.4 + 0.6 x 0.4
= 0.16 + 0.24 = 0.4

ii without replacement

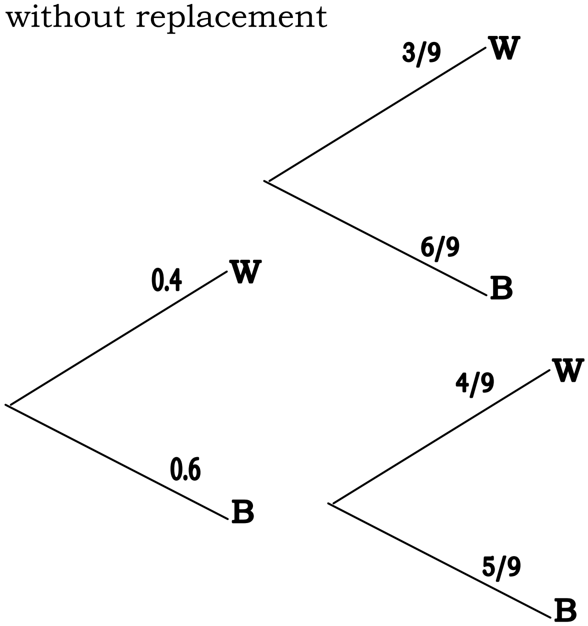
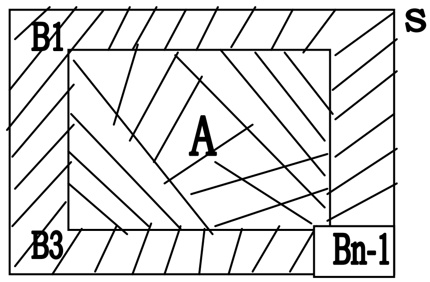


Figure 4.4 Tree Diagram

$$P(2^{nd}\text{ white}) = 0.4 \times 3/9 + 0.6 \times 4/9$$
$$= 0.133 + 0.267 \cong 0.4$$

Let $B_1, B_2, \ldots B_n$ for a partition of a sample space S. Let $A \in S$, then



$$P(A) = P(A/B_1).\ P(B_1) + P(A/B_2)\ P(B_2) + \ldots\ldots + P(A/B_n)\ P(B_n)$$

That is,

$$P(A) = \sum_{i=1}^{n} P(A/B_i).P(B_i) \qquad (4.10)$$

i = 1

4.1 BAYES'S THEOREM

Bayes's theorem was developed by an English Presbyterian minister, Reverend Thomas Bayes (1702 – 1761). This is an expanded form for conditional probabilities.

$$P(B_i/A) = \frac{P(B_i).\ P(A/B_I)}{\sum [\ P(B_i).\ P(A/B_i)]} \qquad (4.11)$$

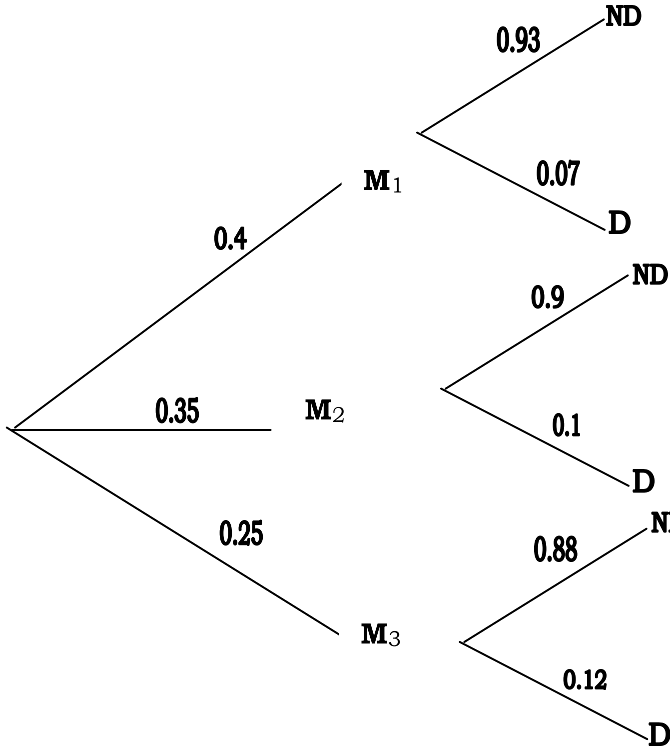
where $B_1, B_2, \ldots.. B_n$ is an all – inclusive set of possible outcomes given A

Example 4.22: A product is being produced by three machines M_1, M_2 and M_3. These machines produce 40%, 35% and 25% of the product respectively. Accordingly, the respective defective products produced by these machines M_1, M_2 and M_3 are 7%, 10% and 12% respectively. Find

i. the probability that a part selected at random from the finished product is defective

ii. the probability of the defective product was produced by machine M_1, M_2, or M_3.

Solution



M_1	-	Machine 1	ND-Non defective product
M_2	-	Machine 2	D-Defective product
M_3	-	Machine 3	

$P(M_1) = 0.4$ $\qquad P(D/M_1) = 0.07$

$P(M_2) = 0.35$ $\qquad P(D/M_2) = 0.1$

$P(M_3) = 0.25$ $\qquad P(D/M_3) = 0.12$

i. $$P(D) = P(M_1)\,P(D/M_1) + P(M_2)\,P(D/M_2) + P(M_3)\,P(D/M_3)$$
$$= 0.4 \times 0.07 + 0.35 \times 0.1 + 0.25 \times 0.12$$
$$= 0.028 + 0.035 + 0.03 = 0.093$$

ii. $$P(M_i/D) = \frac{P(M_i \cap D)}{\sum P(M_i \cap D)} = \frac{P(M_i)\,P(D/M_i)}{\sum P(M_i)\,P(D/M_i)}$$

Thus

$$P(M_i/D) = \frac{0.4 \times 0.07}{0.93} = \frac{0.028}{0.093} \approx 0.3011$$

$$P(M_2/D) = \frac{0.35 \times 0.1}{0.93} = \frac{0.035}{0.093}$$
$$\approx 0.3763$$

$$P(M_3/D) = \frac{0.25 \times 0.12}{0.093} = \frac{0.035}{0.093}$$

$$\approx 0.3226$$