INTRODUCTION TO PROBABILITY

The application of probability is evident in most areas of human endeavour. For example, the chance of an accident occurring on a road, probability of getting a head when a coin is tossed, chance of a top politician winning an election, e.t.c. are examples of probability. Therefore, we must be able to assess the degree of uncertainty, in any given situation, and this is done mathematically by using *probability*

4.1 PROBABILITY OF EVENTS

We begin by defining some terminology that we are using in this chapter and in subsequent ones.

Experiment: Any process that yields a result or an observation.

Outcome: A particular result of an experiment

Sample space: The set of all possible outcomes of an experiment.

Sample point: The individual outcomes in a sample space.

Event: Any subset of the sample space. If A is an event, then n(A) is the number of sample points that belong to event A.

Probability of an event is a measure of the likelihood of that event occurring. If an experiment has a finite number of outcomes which are equally likely, then the probability that an event A will occur is given by

$$P(A) = \frac{\text{number of ways A can occur}}{\text{Total number of possible outcomes}} \qquad (4.1)$$

Example 4.1: A die is tossed once and the outcome could be any of these: The sample space is

$$S = \{ 1, 2, 3, 4, 5, 6 \}$$

Example 4.2: Lets toss a coin twice and the outcome for each of toss in recorded. The sample space is shown here in two different ways.

Diagram Representation

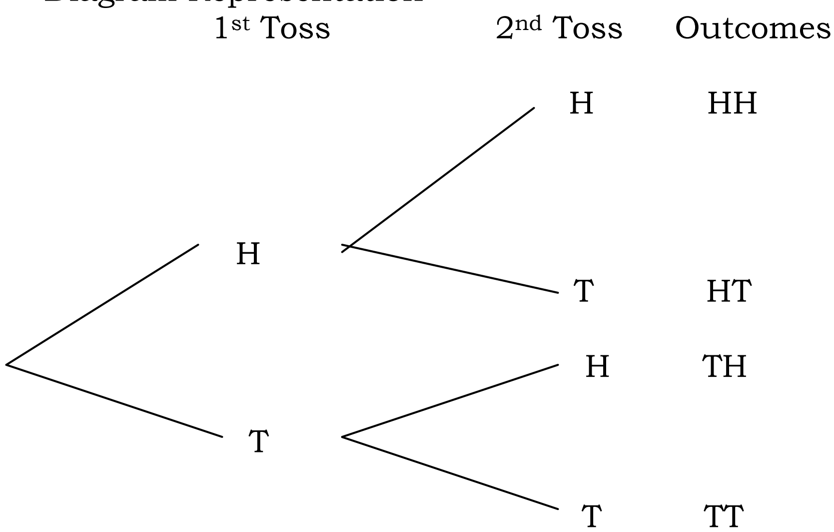

Figure 4.1: Tree Diagram

Listing

$S = \{ HH, HT, TH, TT \}$

$n(S) = 4$

Example 4.3: Lets toss a coin thrice and the outcome for each toss is recorded.

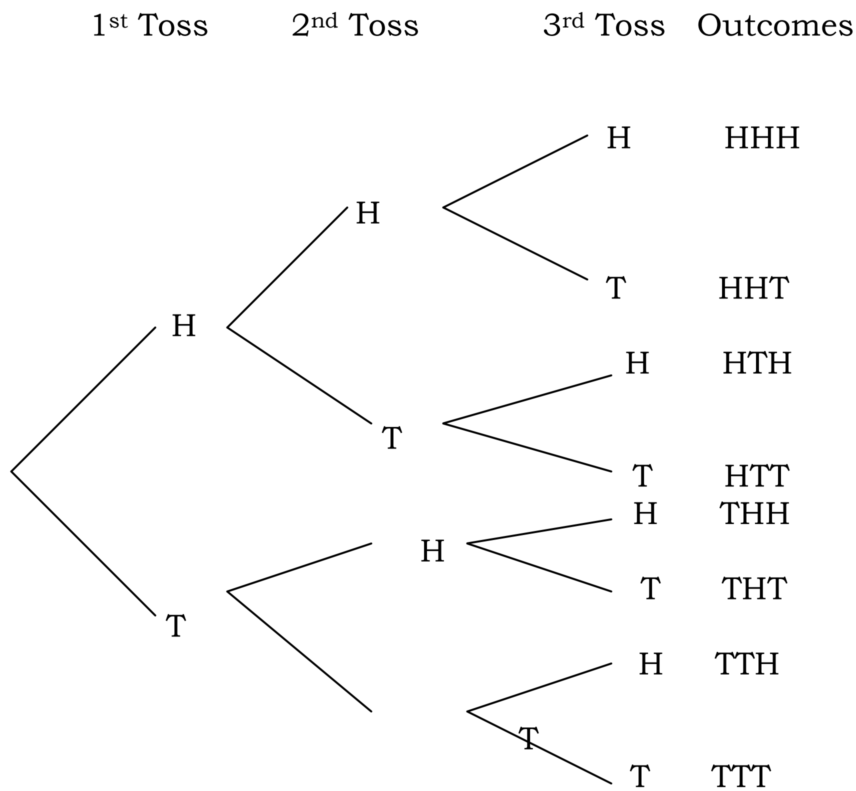

Figure 4.2 Tree Diagram

S = { HHH, HHT, HTH, THH,HTT,THT,TTH,TTT }
n(S) = 8

Example 4.4. Two dice are rolled and the sum of the numbers appearing are observed.

Table 4.1

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

The sample space S = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

x	2	3	4	5	6	7	8	9	10	11	12	
n(x)	1	2	3	4	5	6	5	4	3	2	1	36

with a total of 36-point sample space.

4.2 PERMUTATIONS AND COMBINATONS

4.2.1 Permutation

Permutation is a special arrangement of a group of objects in some order. Any other arrangement of the same objects is a different permutation. The key words for permutation are *order* or *arrangement*. For example, lets arrange n people in order. There are n possible chances for the first person, n-1 remaining possible chances for the second person, n-2 remaining possible chances for the third person, e.t.c, that is,

The number of possible arrangement $= n \times (n - 1) \times (n - 2) \times \ldots . \times 1)$
$= n!$ (n factorial)

Example 4.5

$0! = 1$
$1! = 1$
$2! = 2 \times 1 = 2$
$3! = 3 \times 2 \times 1 = 6$
$4! = 4 \times 3 \times 2 \times 1 = 24$
$5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
$6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 720$

$$nPr = \frac{n!}{(n - r)!}$$

this is the number of permutations of n objects taken r at a time.

Example 4.6: In how many ways can three people be seated on 6 seats in a row?

Solution

Arranging 3 people on 6 seats $= 6P_3$

$$6P_3 = \frac{6!}{(6-3)!} = \frac{6!}{3!} = \frac{6 \times 5 \times 4 \times 3!}{3!}$$

$$= 6 \times 5 \times 4 = 120 \text{ ways}$$

Example 4.7: How many distinct arrangements can be made using all the letters of the word Economics.

Solution

From the word Economics, o = 2, c = 2, and total letters = 9

$$\therefore \text{Total arrangement} = \frac{(\text{Number of letter})!}{(\text{Frequency of letters})!}$$

$$= \frac{9!}{2!\,2!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2}{2 \times 2}$$

$$= 90720$$

Example 4.8: How many different numbers of six digits can be formed using digits 4, 4, 6, 6, 6, 6.

Solution

Total digits (n) = 6
4 has frequency = 2
6 has frequency = 4

$$\text{Total numbers that can be formed} = \frac{5!}{2!\,4!}$$

$$= \frac{6 \times 5 \times 4!}{2 \times 1 \times 4!}$$

$$= 15$$

Example 4.9

A plate number is to be made so that it contains four letters and four digits. Two letters begin the plate number and two letters end it. In how many ways can this number be made so that the first digits is not zero when

i. Both letters and digits cannot be repeated
ii. Both letters and digits can be repeated

Solution

For four letters and four digits, together will result to 8 boxes.

There are 26 alphabets a, b, c, …. z.
There are 10 digits 0, 1, 2, ……, 9.

i. Without Repetition

26	25	9	9	8	7	24	23

number of ways = 26 x 25 x 9 x 9 x 8 x 7 x 24 x 23
= 1,627,516,800 number plates

ii. With Repetition

26	26	9	10	10	10	26	26

number of ways = 26 x 26 x 9 x 10 x 10 x 10 x 26 x 26
= $26^4 \times 10^3 \times 9$ = 4,112,784,000 number plates

4.2.2 Combination

Combination is any collection of a group of objects without regard to order. Problems involving combinations, where order is not relevant, are very similar to problems involving combinations, where order is critical. The only difference between permutations and combinations is whether order matters.

$$nC_r = \frac{n!}{(n-r)!\, r!}$$

is the number of possible combinations of n objects taken r at a time.

Example 4.10: Find the number of ways in which three students can be selected from five students.

Solution

3 students can be chosen from 5 students in $5C_3$ ways

$$= \frac{5!}{(5-3)!\, 3!} = \frac{5!}{2!\, 3!} = \frac{5 \times 4 \times 3!}{2!\, 3!}$$

$$= 5 \times 2 = 10 \text{ ways}$$

Example 4.11

A Mathematics examination consists of 8 questions out of which candidates are to answer 5. In how many ways can each candidate select if

a) There is no compulsory question
b) The first 3 questions are compulsory
c) At least 3 out of the first 4 questions are compulsory

Solution

a. From 8 questions to answer 5 questions, if there is no compulsory question,

we have $8C_5 = \frac{8!}{(8-5)!\, 5!} = \frac{8!}{3!\, 5!}$

$$= \frac{8 \times 7 \times 6 \times 5!}{3 \times 2 \times 5!}$$

$$= 56 \text{ ways}$$

b. If the first 3 questions are compulsory, then a candidate can choose 2 more questions from the remaining 5,

$$5C_2 = \frac{5!}{(5-2)!\ 2!} = \frac{5!}{3!\ 2!} = \frac{5 \times 4 \times 3!}{3!\ 2!}$$

$$= 10 \text{ ways}$$

c. At least 3 out of the first 4 questions are compulsory means the candidate may answer 3 out of the first 4 compulsory questions and 2 from the remaining 4 questions or all the 4 first compulsory questions and 1 from the remaining 4 questions.

$$4C_3 \times 4C_2 + 4C_4 \times 4C_1$$

$$= \frac{4!}{(4-3)!\ 3!} \times \frac{4!}{(4-2)!\ 2!} + \frac{4!}{(4-4)!\ 4!} \times \frac{4!}{(4-1)!\ 1!}$$

$$= \frac{4!}{1!\ 3!} \times \frac{4!}{2!\ 2!} + \frac{4!}{0!\ 4!} \times \frac{4!}{3!\ 1!}$$

$$= 4 \times 6 + 1 \times 4 = 24 + 4 = 28 \text{ ways}$$

Example 4.12: From a gathering of 100 people of which 40 are men, a committee of 15 is to be formed. In how many ways can this be done so that (i) 3 men are there? (ii) no man is included?

Solution

Total number of people = 100
Men = 40, Women = 100 – 40 = 60

i. 3 men in the committee means 12 women in the committee

$$40C_3 \times 60C_{12} = \frac{40!}{(40-3)!\ 3!} \times \frac{60!}{(60-12)!\ 12!}$$

$$= \frac{40!}{37!\ 3!} \times \frac{60!}{48!\ 12!} = \frac{40 \times 39 \times 38}{6} \times \frac{60!}{48!\ 12!}$$

$$= 40 \times 13 \times 19 \times \frac{60!}{48! + 2!} = \frac{9880 \times 60!}{48!\ \ 12!}$$

ii. If no man is included, it means the whole of the committee members are women. We have 60 women in the gathering.

$$60C_{15} \text{ x } 40C_0 = 60C_{15} = \frac{60!}{(60-15)!\ 15!} = \frac{60!}{45!\ 15!}$$

Example 4.13: A bag contains 2 white and 3 red balls. In how many ways can 3 balls be chosen if

a. at least one ball must be white?
b. at least one ball must be red?

Solution

White balls = 2, red balls = 3
Total balls = 5

a. To choose at least one white ball means one white or more is to be chosen. That is, 1W and 2R or 2W and 1R

$= 2C_1 x 3C_2 + 2C_2 x 3C_1$
$= 2x3 + 1x3 = 6 + 3 = 9$ ways

b. to chose at least one red means one red or more. That is, 1R and 2W or 2R and 1W or 3R and no white

$= 3C_1 \text{ x } 2C_2 + 3C_2 \text{ x } 2C_1 + 3C_3$
$= 3 \text{ x } 1 + 3 \text{ x } 2 + 1 = 3 + 6 + 1 = 10$ ways

4.3 LAWS OF PROBABILITY

A probability is always a numerical value between zero and one.

Property I

$$O \le P(A) \le 1$$

Property II

$$\sum_{alloutcomes} P(x) = 1$$

Property 2 states that if we add up the probabilities of each of the sample points in the sample space, the total probability must equal one.

Example 4.14: Find the probability that a head will appear when two coins are tossed.

Solution

The sample space = { HH, HT, TH, TT}
Let event A be the occurrence of one head.

$$P\text{ (a head will appear)} = \frac{\text{numbers of A in sample space}}{\text{number in sample space}}$$

$$= 2/4 = 0.5$$

Example 4.15: Two dice are rolled and the sum of the numbers appearing are observed. Find the possibility of getting (i) a total of 5 (ii) a total of 12.

Solution
From example 4.3, the sample space is given as
(i) $P\text{ (of getting a total of 5)} = \frac{\text{numbers with total of 5}}{\text{number in sample space}}$

$$= \frac{4}{36}$$

$$= 1/9$$

ii. P (of getting a total of 12) = 1/36

COMPLEMENT OF AN EVENT

The set of all sample points in the sample space that do not belong to event A. The complement of event A is denoted by A^1 or A^c.

This also implies that $P(A) + P(A^1) = 1$

Example 4.16: Find the probability that at least a tail will appear when two coins are tossed.

Solution

Let A be the occurrence of no tail then A^1 will be the occurrence of at least one tail will appear.

Sample space = {HH, HT, TH, TT}

$P(A) = ¼$
Therefore, $p(A^1) = 1 – P(A) = 1 – ¼ = ¾$

Combined events are formed by combining several simple events. For example, the probability of either event A or event B will occur is P(A or B) or $P(A \cup B)$; the probability that both events A and B will occur is P(A and B) or $P(A \cap B)$.

MUTUALLY EXCLUSIVE EVENTS

These are events defined in such a way that the occurrence of one event precludes the occurrence of any of the other events. (In short, if one of them happens, the others cannot happen.)

Consider an experiment in which two dice are tossed. Three events are defined.

A: the sum of the numbers on the two dice is 5
B: the sum of the numbers on the two dice is 8.
C: each of the two dice shows the same number.

Table 4.2

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Events A and B are mutually exclusive, because the sum on the two dice cannot be both 5 and 8 at the same time. It is clearly seen in Table 4.2 that events A and B do not intersect at a common sample point. Therefore, they are mutually exclusive. Point (4, 4) satisfies C, and the total of the two 4s satisfies B.

ADDITION LAW

If events A and B are mutually exclusive, then

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) \quad (4.3)$$

i.e. $P(A \cap B) = 0$

If the events A and B are not mutually exclusive then

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B) \quad (4.4)$$

In other words, if two events A and B are mutually exclusive, then we can find the probability that either A or B will occur by simply adding the individual probabilities. If they are not mutually exclusive, then there is some overlap between them. Thus, to find the probability that either A or B will occur, we must subtract the probability of the duplication P(both A and B) from the sum P(A) + P(B).

This can be expanded to consider more than two mutually exclusive events:

$$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C) \quad (4.5)$$

INDEPENDENCE

Two events A and B are *independent* if the probability of second event B is not affected by the occurrence or nonoccurrence of the first event A. If A and B are independent events, then

$$P(\text{Both } A \text{ and } B) = P(A \cap B) = P(A) \times P(B) \quad (4.6)$$

That is, when A and B are independent events, the probability that both hold is just the product of the individual probabilities. This formula can be extended. If A, B, C, ……, Z are independent events then,

$$P(A \text{ and } B \text{ and } C \text{ and } \ldots \text{ and } Z) = P(A \cap B \cap C \cap \ldots \cap Z) \quad (4.7)$$

$$= P(A).\ P(B).\ P(C) \ldots\ldots P(Z)$$

Example 4.17: Let two events A and B be defined on the same sample space. Suppose $P(B) = 0.2$ and $P(A \cup B) = 0.75$. Find P(A) such that

i. A and B are independent
ii. A and B are mutually exclusive

Solution

i. If A and B are independent, then
$P(A \cap B) = P(A)\, P(B)$

Thus $P(A \cup B) = P(A) + P(B) - P(A)\,P(B)$
$0.75 = P(A) + 0.2 - P(A) \times 0.2$
$= 0.2 + 0.8\, P(A)$

$0.8\, P(A) = 0.75 - 0.2 = 0.55$

$$P(A) = \frac{0.55}{0.8}$$
$= 0.6875$

ii. If A and B are mutually exclusive, then
$P(A \cap B) = 0$

Thus $P(A \cup B) = P(A) + P(B)$
$0.75 = P(A) + 0.2$
$P(A) = 0.75 - 0.2 = 0.55$

Example 4.18: Find the probability of getting:
(i) 2 heads (ii) 1 head (iii) no head if a coin is tossed twice.

Solution

P(getting of head) = ½

i. P(getting two heads) = P(1^{st} is head) P(2^{nd} is head)
= ½ x ½ = ¼

ii. P(getting 1 head) = P(1^{st} is head and 2^{nd} is tail) or
P(1^{st} is tail and 2^{nd} is head)
= P(H T) + P(T H)
= ½ x ½ + ½ + ½ = ¼ + ¼ = ½

iii. P(getting no head) = P(1^{st} is tail and 2^{nd} is tail)
= ½ x ½ = ¼