MOUNTAIN TOP UNIVERSITY

E-Courseware

COLLEGE OF BASIC AND APPLIED SCIENCES

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Published By:

Mountain Top University

COURSE GUIDE

COURSE TITLE: Real Analysis I

COURSE CODE: MTH 209

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COURSE OBJECTIVES

GENERAL INTRODUCTION AND COURSE OBJECTIVES

The objective of this course is to introduce students to mathematical analysis of the field of real numbers and their functions. The course will develop a deeper and more rigorous understanding of calculus including defining terms and proving theorems about sequences and series of real numbers, limits of functions, continuity as well as derivatives of real-valued functions. This course also trains the students on how patterns and mathematical ideas can be translated into formal and rigorous writing.

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1 Module One: SET THEORY

1.1 Introduction

The idea of a set is basic to all of mathematics. All mathematical objects and constructions eventually go back to set theory. This shows that set theory is important for the understanding of other concepts in mathematics.

Definition 1.1. By a set A, we mean a well-defined collection of objects such that it can be determined whether or not any particular object is a member of A.

Generally, sets will be denoted by capital letters and small letters to denote the objects (ie the elements) themselves. If a is in set A, we say that a is a member (or an element) of A and write $a \in A$.
Often a set A is denoted by specifying a property p that is uniquely satisfied by each of its elements. In this case, we write

$$A = \{a : a \text{ satisfies property } p\}$$

We shall use the following symbols

$\in$	is an element of
$\notin$	is not an element of
$\forall$	for all
$\exists$	there exists
iff	if and only if

Objectives
At the end of this module, students should be able to:

a. prove basic set theoretic statements and emphasize the proofs' development;

b. identify various kinds of sets;

c. reproduce the formal definitions of operations on sets (set comprehension, subset, intersection, union, complement, set difference, empty set, power set, Cartesian product).

Pre-Test

a. What is a set?

b. List four sets operations

c. What is a countable set?

1.2 Subset and Superset

If every element of a set A is also an element of a set B, then A is called a subset of B and we write $A \subseteq B$. In this case we can also say that B is a superset of A and write $B \supseteq A$. However, if there is an object $b \in B$ which is not in A, then A is a proper subset of B and we write $A \subset B$ or $B \supset A$.

Definition 1.2. A set without an element is called an empty set and is denoted by $\emptyset$.

An empty set is a subset of every set.

Definition 1.3. Assume A and B are sets.

(i) The union of A and B (denoted $A \cup B$) is the set of all objects that belong to A or B or to both A and B.

$$A \cup B = \{a : a \in A \text{ or } a \in B\}$$

(ii) The intersection of A and B (denoted $A \cap B$) is the set of all objects that belong to both A and B.

$$A \cap B = \{a : a \in A \text{ and } a \in B\}.$$

If $A \cap B = \emptyset$, then A and B are said to be disjoint.

(iii) A and B are equal if they have the same elements. $A \subseteq B$ and $B \subseteq A$ iff $A = B$.

(iv) The difference between A and B (denoted by $A \setminus B$) is the set of all objects that belong to A but not in B. If B is a subset of A, then $A \setminus B$ is called the complement of B in A and denoted by B'.

Theorem 1.4. *Let A, B and C be any sets, then*

(i) $A \cap A = A$ *and* $A \cup A = A$ *(idempotence)*

(ii) $A \cap B = B \cap A$ *and* $A \cup B = B \cup A$ *(commutativity)*

(iii) $(A \cap B) \cap C = A \cap (B \cap C)$ *and* $(A \cup B) \cup C = A \cup (B \cup C)$ *(associativity)*

(iv) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *and* $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ *(distributivity)*

Proof. (i), (ii) and (iii) are evident. We prove the distributivity.

(a) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. We need to show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.
Let $x \in A \cap (B \cup C)$ then $x \in A$ and $x \in B \cup C$. This means that $x \in A$, and either $x \in B$ or $x \in C$. Thus we have (i) $x \in A$ and $x \in B$, or we have (ii) $x \in A$ and $x \in C$. Therefore $x \in A \cap B$ or $x \in A \cap C$ so $x \in (A \cap B) \cup (A \cap C)$. Since x was arbitrary, we have $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

Conversely, let $y \in (A \cap B) \cup (A \cap C)$, then $y \in (A \cap B)$ or $y \in (A \cap C)$. It follows that $y \in A$ and either $y \in B$ or $y \in C$. Therefore, $y \in A$ and $y \in B \cup C$, so that $y \in A \cap (B \cup C)$. Since y was arbitrary, we have $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

(b) Left for the students. □

Theorem 1.5. *The sets $A \cap B$ and $A \setminus B$ are non-intersecting and $A = (A \cap B) \cap (A \setminus B)$.*

Proof. Suppose $x \in A \setminus B$, then $x \in A$ and $x \notin B$. Therefore $x \notin A \cap B$. Conversely, suppose $x \in A \cap B$, then $x \in A$ and $x \in B$. Therefore $x \in A \setminus B$.

If $x \in A$, then either $x \in B$ or $x \notin B$. In the case $x \in B$, $x \in A \cap B$. In the latter situation, $x \in A$ and $x \notin B$ so that $x \in A \setminus B$. Thus $x \in (A \cap B) \cup (A \setminus B)$. Since x was arbitrary, we conclude that $A \subseteq (A \cap B) \cup (A \setminus B)$. Conversely, suppose $y \in (A \cap B) \cup (A \setminus B)$, then $y \in A \cap B$ or $y \in A \setminus B$. In either case $y \in A$. Since y was arbitrary, $(A \cap B) \cup (A \setminus B) \subseteq A$. □

Theorem 1.6. *(De Morgan Laws for Three Sets) If A, B, C are any sets, then*

(a) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$

(b) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Proof. (a) We need to show that $A \setminus (B \cup C) \subseteq (A \setminus B) \cap (A \setminus C)$ and $(A \setminus B) \cap (A \setminus C) \subseteq A \setminus (B \cup C)$. Suppose $x \in A \setminus (B \cup C)$, then $x \in A$ but $x \notin (B \cup C)$. Hence $x \in A$ but x is neither in B nor in C. Therefore $x \in A$ but $x \notin B$ and $x \in A$ but $x \notin C$. That is $x \in A \setminus B$ and $x \in A \setminus C$. This shows that $x \in (A \setminus B) \cap (A \setminus C)$.

Conversely, if $x \in (A \setminus B) \cap (A \setminus C)$, then $x \in A \setminus B$ and $x \in A \setminus C$. It follows that $x \in A$ and x is neither in B nor in C. That is $x \in A$ and $x \notin B \cup C$. It follows that $x \in A \setminus (B \cup C)$.

(b) Students should supply this. □

1.3 Cartesian Product

Definition 1.7. If A and B are two non-empty sets, then the cartesian product $A \times B$ of A and B is the set of all ordered pairs (a, b) with $a \in A$ and $b \in B$. The order in which a and b are written is considered essential to the identity of (a, b).

For example, if $A = \{a, b, c\}$ and $\{1, 2\}$ then

$$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}.$$

$A \times B$ can be visualized as the set of six points in the plane where the coordinates are the elements of the set $A \times B$.

Example 1.8. If $A = \{x \in \mathbb{R} : 1 \leq x \leq 2\}$ and $B = \{x \in \mathbb{R} : 1 \leq x \leq 2\} \cup \{x \in \mathbb{R} : 3 \leq x \leq 4\}$, $A \times B$ can be written as

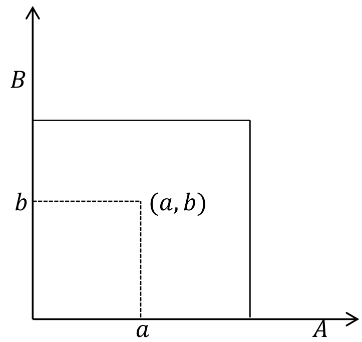

Figure 1.1: Ordered pair (a, b)

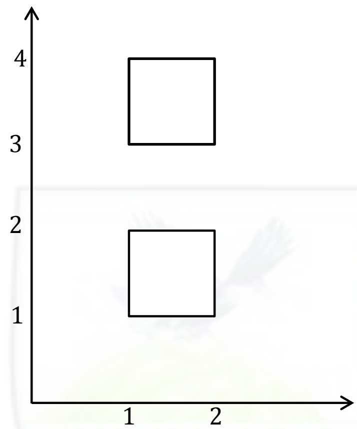

Figure 1.2: The cartesian product $A \times B$

If the cartesian product of $\mathbb{R}$ with itself is formed, the result is the cartesian plane. A nice geometric representation of the cartesian plane can be constructed using two real lines intersecting at a right angle at their origins.
In the sequel, we shall be discussing types of sets.

1.4 Finite and Infinite Sets

In this section, we shall assume familiarity with the set of natural numbers, $\mathbb{N}$.

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

The set of natural numbers is a well-ordered set (ie giving any two different natural numbers x, y, one can tell whether $x < y$ or $x > y$).

Definition 1.9. A set X is finite if it is empty or if there is a one-to-one function with domain X and range in initial segment of $\mathbb{N}$. If there is no such function, the set is infinite. In other words, a finite set is a set which one can count the elements and finish counting. For example,

$\{2, 3, 4, 7\}$.
The number of elements of a finite set X is a non-negative integer and is called the cardinality of the set X. For example if

$$X = \{a, b, i, j, l, m\}$$

$n(X) = 6$.

1.5 Countable and Uncountable Sets

Definition 1.10. A set S is countable if there exists an injective function f from S to the set of natural numbers.

The elements of a countable set can always be counted one at a time and, although the counting may never finish, every element of the set is associated with a unique natural number. For example, the set of all fractions between 0 and 1 is countable because the elements can be ordered

$$\left\{\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \ldots\right\}$$

On the other hand, the set of real numbers is not countable. No matter how the elements are arranged, one cannot provide a one-to-one relationship with the natural numbers such that there is a definite first, second, third elements etc, and still cover every real number.

Remark 1.11. All finite sets are countable, but countable sets could be infinite. For example, the sets of integers and rational numbers are countably infinite. Countably infinite sets are also called denumerable sets.

We now state some results which are obvious.

Theorem 1.12. *(i) Any subset of a finite set is finite.*

(ii) Any subset of a countable set is countable.

(iii) The union of a finite collection of finite sets is a finite set.

(iv) The union of a countable collection of countable sets is countable set.

Post Test

1. If A, B, C are any sets, prove that

$$A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$$

2. If A, B, C are any sets, prove that

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

3. Prove that the subset of a finite set is finite.

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2 Module Two: THE REAL NUMBERS

Objectives

At the end of this module, students should be able to:

a. use field and order axioms of real numbers to establish basic results on real numbers;

b. establish triangle inequality;

c. give the definition of lower and upper bounds;

d. prove various theorems about supremum and infimum.

Pre-Test

a. Define "set of natural numbers";

b. Give two examples of denumerable sets

2.1 Introduction

We think of the real numbers as the points on a line stretching off to infinity in both directions. The real numbers from an algebraic object known as a field, meaning that one can add, subtract and multiply real numbers and divide by non-zero real numbers.

The Field Axioms For all $x, y, z \in \mathbb{R}$,

(i) $x + y = y + x$

(ii) $(x + y) + z = x + (y + z)$

(iii) $\exists\, 0 \in \mathbb{R}$ such that $x + 0 = x \quad \forall x \in \mathbb{R}$

(iv) For each $x \in \mathbb{R}\ \exists\ -x \in \mathbb{R}$ such that $x + (-x) = 0$

(v) $xy = yx$

(vi) $(xy)z = x(yz)$

(vii) $\exists\, 1 \in \mathbb{R}$ such that $1 \neq 0$ and $x \cdot 1 = x\ \forall\, x \in \mathbb{R}$

(viii) if $x \neq 0$, $\exists\, x^{-1} \in \mathbb{R}$ such that $x(x^{-1}) = 1$

(ix) $x(y + z) = xy + yz$

Order Axiom

(i) Tricotomy law: For all $x, y \in \mathbb{R}$, exactly one of the following three relations must hold:

$$\text{(a) } x = y \qquad \text{(b) } x < y \qquad \text{(c) } x > y$$

The tricotomy axiom asserts that every real number other than zero is either positive or negative but never both, and zero is neither positive nor negative.

(ii) For all $x, y, z \in \mathbb{R}$,

$$x < y \text{ if only if } x + z < y + z$$

(iii) For all $x, y \in \mathbb{R}$, if $x > 0$ and $y > 0$ then

$$x + y > 0 \,; \qquad xy > 0$$

(iv) Transitivity order: For all $x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$ then $x < z$.

Remark 2.1. If $x \le y$ and $y \le x$ then $x = y$. This is because $x < y$ and $y < x$ are not possible at the same time by law of tricotomy.

2.2 Absolute Value

If $x \in \mathbb{R}$, we define the absolute value of x, denoted by $|x|$, by

$$|x| = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -x & \text{if } x < 0 \end{cases} .$$

For example, $|2| = 2$, $|-2| = -(-2) = 2$. Intuitively, the absolute value of x represents the distance of x from 0.

Theorem 2.2. *(i)* $|x| \ge 0$, $\forall\, x \in \mathbb{R}$*;* $|x| = 0$ *iff* $x = 0$*;*

(ii) $|x| \le a$ *iff* $-a \le x \le a$*;*

(iii) $|xy| = |x||y|$ $\forall\, x, y \in \mathbb{R}$*;*

(iv) Triangle inequality: $|x + y| \le |x| + |y|$ $\forall\, x, y \in \mathbb{R}$.

Proof. (i) If $x \ge 0$, then $|x| = x \ge 0$. On the other hand, if $x < 0$, then $|x| = -x > 0$. In both cases,$x \ge 0$.

(ii) Suppose $|x| \le a$, by the definition of $|x|$, $x = |x|$ or $x = -|x|$ which implies $-|x| \le x \le |x|$. It follows that $-a \le -|x| \le x \le |x| \le a \Rightarrow -a \le x \le a$. Conversely, suppose $-a \le x \le a$. If $x \ge 0$, then $|x| = x \le a$, if $x < 0$, $|x| = -x \le a$. In both cases, $|x| \le a$.

(iii) The case is obvious for when $x \geq 0$ and $y \geq 0$. Suppose $x \neq 0$ and $y \neq 0$. If $x < 0$ and $y > 0$, then $xy < 0 \Rightarrow |xy| = -xy = |x||y|$. The case $x > 0$ and $y < 0$ is similar. Now let $x < 0$ and $y < 0$ then $xy > 0 \Rightarrow |xy| = (-x)(-y) = |x||y|$.

(iv) There are four possibilities

a. If $x \geq 0$ and $y \geq 0$, then $x + y \geq 0$, so $|x + y| = x + y = |x| + |y|$.

b. If $x \leq 0$ and $y \geq 0$, then $x + y \leq 0$, so $|x + y| = -x + (-y) = |x| + |y|$.

c. If $x \leq 0$ and $y \geq 0$, then $x + y = -|x| + |y|$, so $|x + y| = |-|x| + |y|| \leq |x| + |y|$.

d. If $x \geq 0$ and $y \leq 0$, then $x + y = |x| + (-|y|)$, so $|x + y| = ||x| - |y|| \leq |x| + |y|$. □

Remark 2.3. If x and y are any two real numbers, then

$$|x - y| \geq ||x| - |y|| \qquad \text{and} \qquad |x + y| \geq ||x| - |y||$$

Proof. By triangle inequality,

$$|x| = |x - y + y| \leq |x - y| + |y| \quad \text{which implies} \quad |x - y| \geq |x| - |y| \tag{2.1}$$

Similarly,

$$|y| = |y - x + x| \leq |x - y| + |x| \quad \text{which implies} \quad |x - y| \geq |y| - |x| \tag{2.2}$$

It follows from (2.1) and (2.2) that $|x - y| \geq ||x| - |y||$.
The proof of $|x + y| \geq ||x| - |y||$ is similar and left for the students. □

2.3 Upper and Lower Bound

Definition 2.4. A subset S of real numbers is said to be bounded from below (or simply bounded below) if there exists $\alpha \in \mathbb{R}$ such that

$$\alpha \leq x \qquad \forall\, x \in S.$$

A subset S of real numbers is said to be bounded from above (or simply bounded above) if there exists $\beta \in \mathbb{R}$ such that

$$x \leq \beta \qquad \forall\, x \in S.$$

S is said to be bounded if it is bounded both from above and from below. In this case, there exists $M > 0$ such that

$$|x| \leq M \qquad \forall\, x \in S.$$

Example 2.5. (i) Any finite subset of $\mathbb{R}$ is bounded. For example, $S = \{-3, -2, 0, 1, 4\}$ is bounded. Here, each element of S is greater than or equal to 3 and less than or equal to 4.

(ii) The set $\mathbb{N} = 1, 2, 3, \ldots$ is bounded below, but not bounded above.

(iii) The set $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ is neither bounded below nor above.

(iv) The set $\left\{ \dfrac{2n}{3n-1}, n = 1, 2, 3, \ldots \right\}$ is bounded below by $\dfrac{2}{3}$ and bounded above by 2.

Note A set which is bounded (say above) by α may not contain α. For example, the set $\mu = (0, 2)$ is bounded above by 2 whereas $2 \notin \mu$.

Definition 2.6. Let S be a subset of $\mathbb{R}$ which is bounded above. An upper bound of S is said to be a supremum (or a least upper bound) of S if it is less than any other upper bound of S. Similarly, if S is bounded below, then a lower bound of S is said to be an infimum (greatest lower bound) of S if it is greater than any other lower bound of S. In other words, if L is the supremum of S then

$$s \leq L \qquad \forall\, s \in S$$

and whenever L' is another upper bound for S, then $L' \geq L$. Similarly, if M is the infimum of S, then

$$s \geq M \qquad \forall\, s \in S$$

and whenever M' is another lower bound for S, then $M' \leq M$.

Remark 2.7. It is obvious that suprema and infima may not always exist. However, a finite set will always have a supremum and an infimum.

Example 2.8. (i) If $X = \{8, 6, -2, 4, 1\}$, then $\inf X = -3$ and $\sup X = 8$. In this case, both supremum and infimum belong to X.

(ii) If $Y = \{y : -3 \leq y < 5\}$, then $\inf Y = -3$ and $\sup Y = 5$. Only the infimum belongs to Y.

(iii) If $Z = \{1, 3, 5, 7, \ldots\}$, then $\inf Z = 1$ and $\sup Z$ does not exist.

(iv) If $\left\{ \dfrac{(-1)^n n}{n+1} : n \in \mathbb{N} \right\}$, then $\inf A = -1$ and $\sup A = 1$. Neither 1 nor -1 belong to A.

Proposition 2.9. *Let $A \subset \mathbb{R}$ be nonempty and bounded above. If α and β are least upper bounds of A, then $\alpha = \beta$, that is, the least upper bound of a nonempty subset of $\mathbb{R}$ which is bounded above is unique.*

Proof. Since α is a least upper bound of A and β also an upper bound of A then $\alpha \leq \beta$. Similarly, since β is a least upper bound of A and α also an upper bound of A then $\beta \leq \alpha$. It therefore follows that $\alpha = \beta$. □

Lemma 2.10. *Let S be a subset of real numbers which is bounded. Then the following equation holds:*

$$\sup(-S) = -\inf S \qquad \textit{and equivalently} \qquad \inf(-S) = -\sup(S)$$

Proof. Let M represent the set of all the lower bounds of S. Let $\alpha = \inf S$ then $\alpha \leq s \; \forall \; x \in S$ and $\alpha' \leq \alpha \; \forall \; \alpha' \in M$. It follows that $-\alpha \geq -s \; \forall \; -s \in -S$ and $-\alpha' \geq -\alpha \; \forall \; -\alpha' \in -M$. This implies $-\alpha = \sup(-S)$ ie $\sup(-S) = -\inf S$. □

Lemma 2.11. *Let A and B be bounded subsets of $\mathbb{R}$ such that $A \subseteq B$. Then*

$$\inf B \leq \inf A \leq \sup A \leq \sup B$$

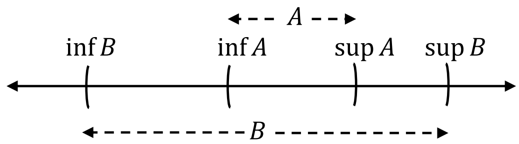

Figure 2.1: $A \subset B$

Proof. We need to prove that

(i) $\inf B \leq \inf A$ (ii) $\inf A \leq \sup A$ (iii) $\sup A \leq \sup B$ (see Figure 2.1)

(i) Let $\alpha = \inf B$ then $\alpha \leq b \; \forall \; b \in B$. Since $A \subseteq B$, $\alpha \leq a \; \forall \; a \in A$. Hence α is a lower bound for A so that $\alpha \leq \inf A$. But $\alpha = \inf B$. Hence $\inf B \leq \inf A$.

(ii) It is clear from the definition of infimum and supremum that $\inf A \leq a \leq A \; \forall \; a \in A$.

(iii) Let $\beta = \sup B$, then $b \leq \beta \; \forall \; b \in B$. Since $A \subseteq B$, $a \leq \beta \; \forall \; a \in A$. Hence β is an upper bound for A, therefore $\sup A \leq \beta$. It therefore follows that $\sup A \leq \sup B$. □

Example 2.12. Let $A = (0, 1)$. Prove that $\sup A = 1$.

Solution Clearly, 1 is an upper bound for A. We claim that $\exists \; 0 < b < 1$ which is also another upper bound for A. Let x be the midpoint of 1 and b, ie $x = \dfrac{1+b}{2}$. It is obvious that $x \in A$, but $x - b = \dfrac{1-b}{2} > 0$. Thus $x > b \; \forall \; x \in A$. Hence b is not an upper bound of A.
Note The empty set is bounded above by any real number, hence it does not have a supremum.

We state (without proof) the fundamental property of the real number system that every non-empty subset of $\mathbb{R}$ which is bounded above has:

Theorem 2.13. *Every non-empty subset of real numbers which has an upper bound also has a supremum.*

Theorem 2.14. *(Nested Interval Theorem) Let $J_n := [a_n, b_n]$ be intervals in $\mathbb{R}$ such that $J_{n+1} \subset J_n \; \forall \; n \in \mathbb{N}$. Then $\bigcap_{n=1}^{\infty} J_n \neq \emptyset$.*

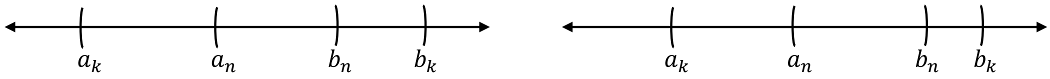

Figure 2.2: Cases where $k \leq n$ and $k > n$ respectively.

Proof. Let A be the set of left endpoints of J_n. Thus, $A := \{a \notin \mathbb{R} : a = a_n \text{ for some } n\}$. It is obvious that A is nonempty. We claim that b_k is an upper bound for A for each $k \in \mathbb{N}$, ie $a_n \leq b_k$ for all n and k. If $k \leq n$, then $[a_n, b_n] \subseteq [a_k, b_k]$ and hence $a_n \leq b_n \leq b_k$. If $k > n$, then $a_n \leq a_k \leq b_k$. Thus the claim is established. By the definition of least upper bound, there exists $c \in \mathbb{R}$ such that $c = \sup A$. We claim that $c \in J_n$ $\forall n \in \mathbb{N}$. Since c is an upper bound for A, we have $a_n < c$ $\forall n$. Since each b_n is an upper bound for A and c is the least upper bound for A, we have $c \leq b_n$. Thus we conclude that $a_n \leq c \leq b_n$ $\forall\, n \in \mathbb{N}$. Hence $c \in J_n$. □

Post Test

1. Let $a, b \in \mathbb{R}$, prove that
$$||a| - |b|| \leq |a + b| \leq |a| + |b|$$

2. If $A, B \subseteq \mathbb{R}$, $A \neq \emptyset$, $B \neq \emptyset$, let $A + B = \{a + b : a \in A, b \in B\}$.

   (a) Show that
   $$\sup(A + B) = \sup A + \sup B$$
   if A and B are bounded above.

   (b) Show that
   $$\inf(A + B) = \inf A + \inf B$$
   if A and B are bounded below.

3. Show that $\sqrt{2}$ is irrational. **Hint:** show that if $\sqrt{2} = \frac{m}{n}$, where m and n are integers, then both m and n must be even. Obtain a contradiction from this.

4. Find the supremum and infimum of each S. State whether they are in S.

   (a) $S = \{x \in \mathbb{R} : x^2 < 4\}$

   (b) $S = \{x \in \mathbb{R} : x^2 < 10\}$

   (c) $S = \{x \in \mathbb{R} : x = \dfrac{2n}{3n+1}, n = 1, 2, \ldots\}$

5. Let X and Y be bounded subsets of $\mathbb{R}$ such that $A \subseteq B$. Then
$$\inf B \leq \inf A \leq \sup A \leq \sup B$$

6. Let $A = (0, 1)$. Prove that

(a) $\sup A = 1$

(b) $\inf A = 0$

7. Let $I_n := [a_n, b_n]$ be intervals in $\mathbb{R}$ such that $I_{n+1} \subseteq I_n \; \forall \, n \in \mathbb{N}$. Prove that

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset$$

8. Show that any nonempty subset of $\mathbb{Z}$ which is bounded above in $\mathbb{R}$ has a supremum.

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3 Module Three: SEQUENCES

Objectives

At the end of this module, students should be able to:

a. define and recognize the sequence of real numbers and convergence ;

b. establish the convergence or divergence of sequences of real numbers;

c. give the relationship between convergence and boundedness of real sequences;

d. prove various theorems about convergence of real sequences;

e. establish basic results about monotone sequences.

Pre-Test

a. What is a sequence?

b. What type of set is a sequence?

3.1 Introduction

Definition 3.1. Let X be a nonempty set. A sequence in X is a function f on an infinite subset of $\mathbb{N}$ the set of natural numbers whose range is contained in the set X. We let $x_n := f(n)$ and call x_n the n-th term of the sequence. One usually denotes f by $\{x_n\}$ or as an infinite tuple $(x_1, x_2, \ldots, x_n, \ldots)$.

In this course we denote a sequence by $\{x_n\}_{n=1}^{\infty}$ (or simply $\{x_n\}$) and n-th term of the sequence by x_n. We shall be concerned with sequences in $\mathbb{R}$, also called real sequences.

Example 3.2. (i) Let $x_n = \dfrac{1}{n}$, the sequence is

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$$

Another way of writing this sequence is $\left\{\dfrac{1}{n}\right\}$.

(ii) $\left\{\dfrac{(-1)^{n+1}}{n}\right\} = 1, -\dfrac{1}{2}, \dfrac{1}{3}, -\dfrac{1}{4}, \dfrac{1}{5}, \ldots, \dfrac{1}{2k-1}, -\dfrac{1}{2k}, \ldots$

(iii) Let $x_n = \dfrac{n}{n+1} \quad \forall\, n \in \mathbb{N}$

1	2	$\ldots$	100	$\ldots$	1000	$\ldots$	100,000	$\ldots$
$\frac{1}{2}$	$\frac{2}{3}$	$\ldots$	$\frac{100}{101}$	$\ldots$	$\frac{1000}{1001}$	$\ldots$	$\frac{100,000}{100,001}$	$\ldots$

It is clear from the above table that as n increases, the value of x_n approaches the value 1. ie

$$\lim_{n\to\infty} x_n = 1 \qquad \text{or} \qquad \lim x_n = 1 \qquad \text{or} \qquad x_n \to 1 \text{ as } n \to \infty$$

Definition 3.3. A sequence $\{x_n\}$ of real numbers is said to converge to a real number x iff for any given $\varepsilon > 0$, there exists a natural number n_0 (which depends on ε) such that

$$|x_n - x| < \varepsilon \qquad \forall\, n \geq n_0$$

The number x is called a limit of the sequence $\{x_n\}$. We then write $\lim_{n\to\infty} x_n = x$. A real sequence which does not converge to a real number is said to diverge.

Remark 3.4. If $f : \mathbb{N} \to \mathbb{R}$ is a sequence, denoted by $\{x_n\}_{n=1}^{\infty}$, its restriction to the subset $\{k \in \mathbb{N} : k \geq n_0\}$ is denoted by $\{x_n\}_{n=n_0}^{\infty}$. $\{x_n\}_{n=n_0}^{\infty}$ is called the tail of the sequence $\{x_n\}_{n=1}^{\infty}$. Thus if $\{x_n\}$ converges to x, then we want the entire tail starting from n_0 to lie in the interval $(x - \varepsilon, x + \varepsilon)$.

Example 3.5. (i) Show that $\left\{\frac{1}{n}\right\} \to 0$ as $n \to \infty$

(ii) Show that $\left\{\frac{1}{2^n}\right\} \to 0$ as $n \to \infty$

(iii) Show that $\lim_{n\to\infty} \left(\frac{3n+1}{7n-4}\right) = \frac{3}{7}$

Lemma 3.6. *(Uniqueness of limit) If $x_n \to x$ and $x_n \to y$, then $x = y$.*

Proof. Let $\varepsilon > 0$ be given. Since $x_n \to x$ and $x_n \to y$, there exist integers n_1 and n_2 such that

$$\begin{aligned} |x_k - x| &< \frac{\varepsilon}{2} \qquad \text{for} \quad k \geq n_1 \\ |x_k - y| &< \frac{\varepsilon}{2} \qquad \text{for} \quad k \geq n_2 \end{aligned}$$

Let $n_0 = \max\{n_1, n_2\}$. Then $\forall\, k \geq n_0$, we have

$$|x - y| = |x - x_k + x_k - y| \leq |x_k - x| + |x_k - y| < \varepsilon$$ □

The following lemma gives the behaviour of the absolute values of the terms of a converging sequence

Lemma 3.7. *Let $\{x_n\}$ be a sequence of real numbers.*

(i) If $x_n \to x$, then $|x_n| \to |x|$. However the converse is not true

(ii) The sequence $x_n \to 0$ iff $|x_n| \to 0$

(iii) The sequence $x_n \to x$ iff $(x_n - x) \to 0$ iff $|x_n - x| \to 0$

We are now ready to prove the basic property of convergent sequences

Theorem 3.8. *Every convergent real sequence is bounded.*

Proof. Let $x_n \to x$ and, for simplicity, let $\varepsilon = 1$. There exists $n_0 \in \mathbb{N}$ such that

$$|x_n - x| < 1 \qquad \text{for} \quad n \geq n_0.$$

Then

$$|x_n| \leq |x_n - x| + |x| < 1 + |x| \qquad \text{for} \quad n \geq n_0.$$

This implies $\{x_n\}$ is bounded for $n \geq n_0$. To obtain an estimate for all $\{x_n\}$, set

$$C = \max\left\{|x_1|, |x_2|, \ldots, |x_{n_0-1}|, 1 + |x|\right\}.$$

Then we have

$$|x_n| \leq C \qquad \text{for} \quad n = 1, 2, 3, \ldots$$

□

Remark 3.9. The converse of Theorem 3.8 is not true in general. For example, the sequence $\{(-1)^n\}$ is bounded but not convergent.

Given any two convergent sequences $\{x_n\}$ and $\{y_n\}$. New sequences can be obtained from these by adding or subtracting the terms, by multiplying by scalars or by multiplying term-wisely. The following result address what happens to the newly formed sequence.

Theorem 3.10. *(Algebra of convergent sequences) Let $x_n \to x$, $x_n \to y$ and $\alpha \in \mathbb{R}$. Then*

(i) $x_n + y_n \to x + y$;

(ii) $\alpha x_n \to \alpha x$;

(iii) $x_n y_n \to xy$;

(iv) $\dfrac{1}{x_n} \to \dfrac{1}{x}$ *provided that* $x \neq 0$.

Proof. (i) Given $\varepsilon > 0$. Since $x_n \to x$ $\exists$ $n_1 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{2} \qquad \text{for} \quad n \geq n_1$$

Since $y_n \to y$ $\exists$ $n_2 \in \mathbb{N}$ such that

$$|y_n - y| < \frac{\varepsilon}{2} \qquad \text{for} \quad n \geq n_2.$$

Now, let $n_0 = \max\{n_1, n_2\}$. For $n \geq n_0$

$$|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \varepsilon.$$

(ii) Let $\varepsilon > 0$ be given. Since $x_n \to x$ $\exists$ $n_1 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{|\alpha|} \qquad \text{for} \quad n \geq n_1.$$

Now,

$$|\alpha x_n - \alpha x| = |\alpha||x_n - x| < \varepsilon.$$

(iii) Every convergent sequence is bounded. Therefore $\exists\, C > 0$ such that

$$|x_n| \leq C \qquad \forall\, n \in \mathbb{N}.$$

Let $\varepsilon > 0$ be given. Since $x_n \to x$ and $y_n \to y$, $\exists\, n_1 \in \mathbb{N}$ and $n_2 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon}{2(|y|+1)} \quad \forall\, n \geq n_1 \qquad \text{and} \qquad |y_n - y| < \frac{\varepsilon}{2C} \quad \forall\, n \geq n_2\,.$$

Choose $n_0 = \max\{n_1, n_2\}$. Then $\forall\, n \geq n_0$, we have

$$\begin{aligned}
|x_n y_n - xy| &= |x_n y_n - x_n y + x_n y - xy| \\
&= |x_n(y_n - y) + y(x_n - x)| \\
&\leq |x_n||y_n - y| + |y||x_n - x| \\
&< C\frac{\varepsilon}{2C} + (|y|+1)\frac{\varepsilon}{2(|y|+1)} \quad = \varepsilon
\end{aligned}$$

(iv) Let $\varepsilon > 0$ be given. Since $x_n \to x$ and $x \neq 0$, $\exists\, n_1 \in \mathbb{N}$ such that

$$|x_n| > \frac{|x|}{2} \qquad \text{for} \quad n \geq n_1. \tag{3.1}$$

To see this, observe that

$$|x| \leq |x_n - x| + |x_n| < \varepsilon + |x_n| \qquad \Rightarrow \qquad |x_n| > |x| - \varepsilon.$$

There is an $n_1 \in \mathbb{N}$ such that $\varepsilon = \dfrac{|x|}{2}$. And so (3.1) follows.
Now, there exists $n_2 \in \mathbb{N}$ such that

$$|x_n - x| < \frac{\varepsilon |x|^2}{2} \qquad \forall\, n \geq n_2.$$

Choose $n_0 = \max\{n_1, n_2\}$. Then for every $n \geq n_0$,

$$\begin{aligned}
\left|\frac{1}{x_n} - \frac{1}{x}\right| &= \left|\frac{x - x_n}{x x_n}\right| = \frac{1}{|x_n|}\frac{1}{|x|}|x - x_n| \\
&< \frac{2}{|x|}\frac{1}{|x|} \times \frac{\varepsilon |x|^2}{|2|} = \varepsilon
\end{aligned}$$

□

The result below shows that the order "≤" is preserved when taking limits.

Theorem 3.11. *Suppose $\{x_n\}$ and $\{y_n\}$ are convergent sequences and that $x_n \to x$, $y_n \to y$. If $x_n \leq y_n\ \forall\, n \in \mathbb{N}$, then $x \leq y$.*

Proof. We prove this by contradiction. Suppose $x > y$. Take $\varepsilon = \dfrac{x-y}{2} > 0$. Since $x_n \to x$, there exists $n_1 \in \mathbb{N}$ such that $x - \varepsilon < x_n < x + \varepsilon\ \forall\, n \geq n_1$. Similarly there exists $n_2 \in \mathbb{N}$ such that $y - \varepsilon < y_n < y + \varepsilon\ \forall\, n \geq n_2$. Let $n_0 = \max\{n_1, n_2\}$, then for all $n \geq n_0$,

$$y_n < y + \varepsilon = y + \frac{x-y}{2} = x - \varepsilon < x_n.$$

This contradicts the assumption that $x_n \leq y_n\ \forall\, n \in \mathbb{N}$. Hence $x \leq y$. □

3.2 Monotone Sequences

Definition 3.12. A sequence $\{x_n\}$ of real numbers is called

(i) monotone non-decreasing if $x_{n+1} \geq x_n \quad \forall\, n \in \mathbb{N}$;

(ii) strictly increasing if $x_{n+1} > x_n \quad \forall\, n \in \mathbb{N}$;

(iii) monotone non-increasing if $x_{n+1} \leq x_n \quad \forall\, n \in \mathbb{N}$;

(iv) strictly decreasing if $x_{n+1} < x_n \quad \forall\, n \in \mathbb{N}$.

Remark 3.13. Any non-decreasing sequence is bounded below by x_1. Hence such a sequence is bounded iff it is bounded above. Similarly any non-increasing sequence is bounded above by x_1. Hence such a sequence is bounded iff it is bounded below.

There are four ways to verify that a given sequence $\{x_n\}$ is monotone.

a. Examine the difference $x_{n+1} - x_n$. If $x_{n+1} - x_n \geq 0$ for all $n \in \mathbb{N}$, then the sequence is monotone non-decreasing. If $x_{n+1} - x_n \leq 0$, the sequence is monotone non-increasing.

b. Examine the quotient $\dfrac{x_{n+1}}{x_n}$ (provided $x_n > 0\ \forall\, n \in \mathbb{N}$). If $\dfrac{x_{n+1}}{x_n} \geq 1$ for all $n \in \mathbb{N}$, then the sequence is monotone non-decreasing. If $\dfrac{x_{n+1}}{x_n} \leq 1$, the sequence is monotone non-increasing.

c. The use of calculus. In this case, the sequence is written as a function f of x such that $x \geq 1$. Obtain $f'(x)$ the derivative of f with respect to x, where $x \geq 1$. If $f'(x) \geq 0$ then the sequence is monotone non-decreasing. If $f'(x) \leq 0$, the sequence is monotone non-increasing.

d. Mathematical induction.

Example 3.14. Verify if the sequences defined by

1. $\left\{1 - \dfrac{1}{n}\right\}$

2. $\left\{\dfrac{n}{n+1}\right\}$

are monotone non-decreasing, monotone non-decreasing or not.

A real sequence $\{x_n\}$ that is bounded above has a supremum (say β). If the sequence is non-decreasing, then all its elements will come closer to β but will never exceed β as $n \to \infty$ This suggests that the sequence converges to β. Similarly, a monotone non-increasing sequence of real numbers which is bounded below converges to its infimum. We put these together in the theorem below.

Theorem 3.15.

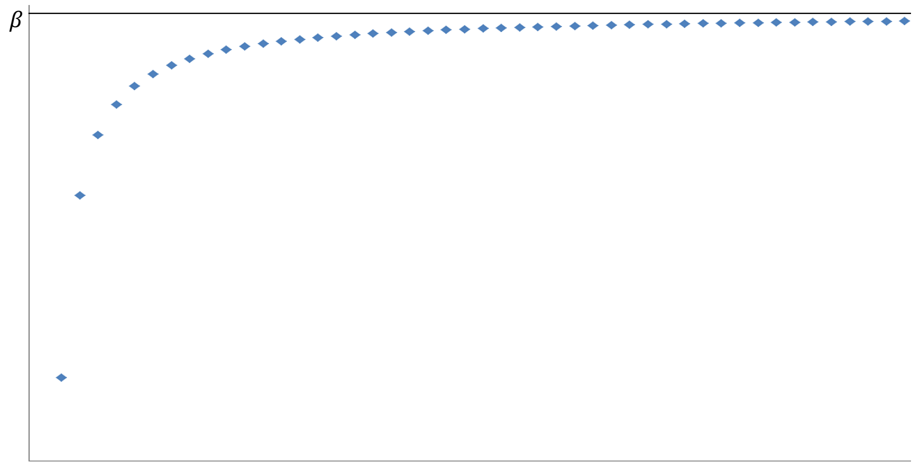

Figure 3.1: Non-decreasing sequence that is bounded above.

(i) A monotone non-decreasing sequence of real numbers which is bounded above converges.

(ii) A monotone non-increasing sequence of real numbers which is bounded below converges.

Proof. We □

Theorem 3.15 can be used to establish the existence of the Euler number e.

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n+1} = \mathrm{e}$$

Lemma 3.16. *Let*

3.3 Completeness of Real Numbers

Definition 3.17. A sequence $\{x_n\}$ in $\mathbb{R}$ is called a Cauchy sequence (or a fundamental sequence) if for each $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$|x_n - x_m| < \varepsilon \qquad \forall\, m, n \geq n_0$$

Definition 3.18. A set S is called complete if every Cauchy sequence in S converges to an element of S.

Theorem 3.19. *(Completeness of $\mathbb{R}$) A real sequence $\{x_n\}$ is Cauchy if and only if it is convergent.*

Proof. We need to show that

1. every convergent real sequence is Cauchy
2. every Cauchy sequence is convergent

□

Remark 3.20. The set $\mathbb{Q}$ of rational numbers is not complete.

Proof. To justify this, it is enough to produce a Cauchy sequence in $\mathbb{Q}$ which converges to an element not in $\mathbb{Q}$. For this, we consider the sequence $\{x_n\}$ defined by $x_n = \left(1 + \frac{1}{n}\right)^n$, $n = 1, 2, 3, \ldots$. This sequence is Cauchy. To see this,

$$x_n - x_m = \left(1 + \frac{1}{n}\right)^n - \left(1 + \frac{1}{m}\right)^m$$

Suppose $n = m + k$ for some $k \in \mathbb{N}$, then

$$\begin{aligned} x_n - x_m &= \left(1 + \frac{1}{m+k}\right)^{m+k} - \left(1 + \frac{1}{m}\right)^m \\ &= \vdots \\ &\leq 4\left[\left(1 + \frac{1}{m}\right)^k - 1\right] \end{aligned}$$

Set $4\left[\left(1 + \frac{1}{m}\right)^k - 1\right] < \varepsilon \quad \Rightarrow \quad m > \frac{1}{\sqrt[k]{\frac{\varepsilon}{4} + 1} - 1}$.

Take $n_0 = 1 + \frac{1}{\sqrt[k]{\frac{\varepsilon}{4} + 1} - 1}$, then

$$|x_n - x_m < |\varepsilon \qquad \text{for} \quad n, m \geq n_0$$

This shows that ∎

Post Test

1. Show that $\frac{2n}{3n+1} \to \frac{2}{3}$ as $n \to 0$.
2. Let $x_n \to x$, $y_n \to y$ and $\beta \in \mathbb{R} \setminus \{0\}$. Prove that
   (a) $x_n - y_n \to x - y$
   (b) $\frac{1}{\beta} x_n \to \frac{1}{\beta} x$
3. True or false. If $\{x_n\}$ and $\{y_n\}$ are sequences such that $x_n y_n \to 0$, then either $x_n \to 0$ or $y_n \to 0$.
4. Let $b_n \geq 0$ and $b_n \to 0$. Assume that there exists an integer N such that $|a_n - a| < b_n$ for all $n \geq N$. Prove that $a_n \to a$.
5. Give a counterexample to show that the set $\mathbb{Q}$ of rational number is not complete.
6. Suppose that $\{a_n\}$ is a convergent sequence with $a \leq a_n \leq b$ for all $n \in \mathbb{N}$. Prove that

$$a \leq \lim_{n \to \infty} a_n \leq b.$$

7. Let $\{x_n\}$ be a sequence such that

$$|x_{n+1} - x_n| \leq \tau |x_n - x_{n-1}|,$$

   for some constant $\tau \in (0, 1)$. Show that $\{x_n\}$ is convergent.

8. Let $\{x_n\}$ be an increasing sequence. Prove that $\{x_n\}$ is convergent if and only if it is bounded above.

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4 Module Four: INFINITE SERIES

Objectives

At the end of this module, students should be able to:

a. define and recognize the series of real numbers as sequences of partial sums;

b. establish the convergence or divergence of infinite series;

c. prove various theorems about convergence of series;

d. test for the convergence of infinite series.

Pre-Test

a. What is geometric series?

b. What is sum to infinity?

4.1 Introduction

Definition 4.1. A series $\sum_{k=1}^{\infty} a_k$ of real numbers is defined as a double sequence $\{a_n, S_n\}$ which satisfies the following condition

$$S_n = \sum_{k=1}^{n} a_k$$

where $a_n - S_n - S_{n-1}$.

The number a_n is called the general term of the series and S_n is called the nth partial sum of the series.

Definition 4.2. We say that the infinite series $\sum_{j=1}^{\infty} a_j$ is convergent if the sequence $\{S_n\}$ of partial sums is convergent. In such a case, the limit $s := \lim S_n$ is called the sum of the series and we denote this fact by the symbol $\sum_{j=1}^{\infty} a_j = s$. We say that the series $\sum_{j=1}^{\infty} a_j$ is divergent if the sequence of its partial sums is divergent.

Definition 4.3. If $s = \sum_{j=1}^{\infty} a_j$, and $\sum_{j=1}^{n} a_j = S_n$, the number $s - S_n = \sum_{j=n+1}^{\infty} a_j$ is called the remainder of the series or the tail end or simply the tail of the series.

Example 4.4. (i) The geometric series $\sum_{j=1}^{\infty} \left(\frac{2}{3}\right)^j = s$ is convergent to the number 2. To see this, observe that

$$\begin{aligned} S_n &= \sum_{j=1}^{n} \left(\frac{2}{3}\right)^j \\ &= \frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots + \left(\frac{2}{3}\right)^n \\ &= \frac{2}{3}\left[\frac{1-\left(\frac{2}{3}\right)^n}{1-\frac{2}{3}}\right] \\ &= 2\left[1-\left(\frac{2}{3}\right)^n\right] \to 2 \quad \text{as} \quad n \to \infty \end{aligned}$$

(ii) Determine whether or not the series $1 + 3 + 5 + \ldots$ converges.

Solution

(iii) Determine whether or not the series

$$\frac{1}{2 \cdot 5} + \frac{1}{5 \cdot 8} + \frac{1}{8 \cdot 11} + \ldots$$

converges.

Solution

The nth term of the given series is $a_n = \dfrac{1}{(3n-1)(3n+2)}$.

$$\begin{aligned} \therefore\ S_n &= \sum_{r=1}^{n} \frac{1}{(3r-1)(3r+2)} \\ &= \frac{1}{3}\sum_{r=1}^{n}\left(\frac{1}{3r-1} - \frac{1}{3r+2}\right) \qquad \text{By partial fractions} \end{aligned}$$

Now, take

Remark 4.5. The method of summation used in the previous example is called telescoping method. The method is particularly suitable in some cases when the general term of a series can be resolved into partial fractions.

Proposition 4.6. *If* $\sum_{r=1}^{\infty} a_r$ *converges, then* $a_r \to 0$.

Proof. Let $\sum_{r=1}^{\infty} a_r$ be convergent. It follows that $\{S_n\}$ is convergent. So let

$$\lim_{n\to\infty} S_n = s.$$

This also implies that $\lim_{n\to\infty} S_{n+1} = s$. By definition,

$$a_n = S_n - S_{n-1},$$

$$\therefore \lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = s - s = 0$$

□

Note The converse of this proposition is not true.

Theorem 4.7. *(Cauchy criterion) The series* $\sum_{n=1}^{\infty} a_n$ *converges iff for each* $\varepsilon > 0$*, there exists* $n_0 \in \mathbb{N}$ *such that*

$$m, n \geq n_0 \qquad \Rightarrow \qquad |S_n - S_m| < \varepsilon.$$

In other words, the series $\sum_{n=1}^{\infty} a_n$ converges iff for each $\varepsilon > 0$, $\exists\, n_0 \in \mathbb{N}$ such that

$$n > m \geq n_0 \qquad \Rightarrow \qquad |a_{m+1} + a_{m+2} + \ldots + a_n| < \varepsilon.$$

Proof. Let $\sum_{n=1}^{\infty} a_n$ be convergent. Then the □

Note: The Cauchy criterion is quite useful when we want to show that a series is convergent without bothering to know its sum.

Given any two series $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ and a scalar $\lambda \in \mathbb{R}$,

$$\sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n := \sum_{n=1}^{\infty} (a_n + b_n)$$
$$\lambda \sum_{n=1}^{\infty} a_n := \sum_{n=1}^{\infty} (\lambda a_n)$$

Theorem 4.8. *(Algebra of convergent series) Let* $\sum_{n=1}^{\infty} a_n$ *and* $\sum_{n=1}^{\infty} b_n$ *be two convergent series with their respective sums* A *and* B*, and let* $\lambda \in \mathbb{R}$*.*

(i) Their sum $\sum_{n=1}^{\infty} (a_n + b_n)$ *is convergent and we have* $\sum_{n=1}^{\infty} (a_n + b_n) = A + B$*.*

(ii) The series $\lambda \sum_{n=1}^{\infty} a_n$ *is convergent and we have* $\lambda \sum_{n=1}^{\infty} a_n = \lambda A$*.*

Proof. (i) Let

(ii)

□

Definition 4.9. The series $\sum_{k=1}^{\infty} a_k$ is said to be absolutely convergent if the infinite series $\sum_{k=1}^{\infty} |a_k|$ is convergent. If a series is convergent but not absolutely convergent, then it is said to be conditionally convergent.

Proposition 4.10. *If* $\sum_{k=1}^{\infty} a_k$ *is absolutely convergent then* $\sum_{k=1}^{\infty} a_k$ *is convergent.*

Proof. Let S_n and σ_n denote the partial sums of $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} |a_k|$ respectively. It is enough to show that $\{S_n\}$ is Cauchy. We have, for $n > m$,

$$|S_n - S_m| = \left| \sum_{k=m+1}^{n} a_k \right| \leq \sum_{k=m+1}^{n} |a_k| = \sigma_n - \sigma_m \ ,$$

which converges to zero because $\{\sigma_n\}$ is convergent. Hence $\{S_n\}$ is a Cauchy sequence. □

Note: The converse of Proposition 4.10 is not true.

4.2 Test of Convergence of Series

How can one know whether or not a series is convergent?

(a) Comparison test

(b) Integral test

(c) d'Alembert's ratio test

(d) Cauchy's root test

(e) The limit comparison test

4.2.1 Comparison Test

Theorem 4.11. *Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be series of non-negative real numbers. Assume that $a_n \leq b_n$ for all $n \in \mathbb{N}$, then*

(i) if $\sum_{n=1}^{\infty} b_n$ is convergent, then so is $\sum_{n=1}^{\infty} a_n$

(ii) if $\sum_{n=1}^{\infty} a_n$ is divergent, so is $\sum_{n=1}^{\infty} b_n$

Proof. Let

$$S_n = \sum_{j=1}^{n} a_j \qquad \text{and} \qquad T_n = \sum_{j=1}^{n} b_j \ .$$

Since $a_k \leq b_k$ for all $k \in \mathbb{N}$, we see that $S_n \leq T_n$.

(i) If $\sum_{n=1}^{\infty} b_n$ is convergent, let $T_n \to t$. We know that $t = \text{lub}\{T_n : n \in \mathbb{N}\}$. Hence $S_n \leq T_n \leq t$ so that t is an upper bound of the set $\{S_n : n \in \mathbb{N}\}$. $\{S_n\}$ is an increasing sequence and bounded above, therefore converges (by Theorem 3.15).

(ii) If $\sum_{n=1}^{\infty} a_n$ is divergent, note that its partial sums form an increasing unbounded sequence. Given $M \in \mathbb{R}$, $\exists\, n_0 \in \mathbb{N}$ such that for $k \geq n_0$, we have $S_k > M$. Hence $T_k \geq S_k > M$ for such k. We therefore conclude that $\sum_{n=1}^{\infty} b_n$ is divergent. □

Example 4.12. Show that comparison that the series $\sum_{n=1}^{\infty} \left(\frac{2n-1}{n}\right)$ is divergent.

solution

It is

4.2.2 Integral Test

Definition 4.13. An infinite integral $\int_b^{\infty} f(t)\, dt$ is said to converge if $\int_b^{N} f(t)\, dt$ tends to a finite number as $N \to \infty$. Otherwise the integral is said to diverge.

If $f : [a, b] \to \mathbb{R}$ is continuous with $\alpha \leq f(x) \leq \beta$ for $x \in [a, b]$, then

$$\alpha(b-a) \leq \int_a^b f(x)\, dx \leq \beta(b-a)\ . \tag{4.1}$$

This inequality can be seen geometrically by considering a non-negative function f and using the geometric interpretation of the definite integral.

Theorem 4.14. *(Integral test) Assume that $f : [1, \infty] \to [0, \infty)$ is continuous and decreasing. Let $a_n := f(n)$ and $b_n := \int_1^n f(t)\, dt$. Then*

(i) $\sum_{n=1}^{\infty} a_n$ converges if $\{b_n\}$ converges,

(ii) $\sum_{n=1}^{\infty} a_n$ diverges if $\{b_n\}$ diverges.

Proof. Since f is decreasing, we have

$$f(n) \leq t \leq f(n-1) \qquad \text{for} \quad t \in [n-1, n],\ n = 2, 3, 4, \ldots$$

It follows from (4.1), that

$$a_n \leq \int_{n-1}^{n} f(t)\, dt \leq a_{n-1}$$

so that

$$\sum_{k=2}^{n} a_k \leq \int_1^n f(t)\, dt \leq \sum_{k=1}^{n-1} a_k\ .$$

This implies

$$\sum_{k=2}^{n} a_k \leq b_n \leq \sum_{k=1}^{n-1} a_k\ .$$

(i) and (ii) follow by comparison (Theorem 4.11). □

Example 4.15. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Solution Let $f(t) = \frac{1}{t^p}, \quad t \geq 1$. Obviously, f is non-negative, monotone decreasing and integrable.
If $p = 1$

4.2.3 d'Alembert's Ratio Test

Theorem 4.16. *Let $\sum_{r=1}^{\infty} a_r$ be a series of positive reals. Assume that*

$$\lim_{n\to\infty} \frac{a_{r+1}}{a_r} = \gamma.$$

Then the series $\sum_{r=1}^{\infty} a_r$ is

1. *convergent if $0 \leq \gamma < 1$*

2. *digergent if $\gamma > 1$.*

The test is inconclusive if $\gamma = 1$.

Proof. If □

Example 4.17. Test the convergence of otherwise of each of the following series

1. $\sum_{n=1}^{\infty} \frac{10^n}{n}$

2. $\sum_{n=1}^{\infty} \frac{a^n}{n^2}, \quad a > 0$

4.3 Rearrangements

Given a series $\sum_{n=1}^{\infty} a_n$, we can obtain another series by rearranging the terms in the series. The following information explains what happens to the new series.

Theorem 4.18. *If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum_{n=1}^{\infty} b_n$ also converges absolutely. Furthermore*

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$$

Remark 4.19. If we rearrange infinitely many terms of a series that converges conditionally, we can get results that are far different from the sum of the original series.

Post Test

1. Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be two convergent series with their respective sums A and B, and let $\lambda \in \mathbb{R} \setminus \{0\}$. Prove that

   a. $\sum_{n=1}^{\infty}(a_n - b_n) = A - B$.

   b. $\frac{1}{\lambda}\sum_{n=1}^{\infty} a_n = \frac{1}{\lambda}A$.

2. Prove that a necessary condition for a series $\sum_{n=1}^{\infty} a_n$ to be convergent is that $\lim_{n\to\infty} a_n = 0$.

3. Show that the geometric series $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ is convergent to the number 2.

4. Use telescoping method to determine whether or not the series

$$\frac{1}{2\cdot 5} + \frac{1}{5\cdot 8} + \frac{1}{8\cdot 11} + \dots$$

   converges.

5. (a) When do we say a series $\sum_{r=1}^{\infty} a_r$ is absolutely convergent?

   (b) Show that if $\sum_{r=1}^{\infty} a_r$ is absolutely convergent, then $\sum_{r=1}^{\infty} a_r$ is convergent.

6. Use integral test to show that the series $\sum_{r=1}^{\infty} \frac{1}{r^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

7. Use comparison test to show that the series $\sum_{r=1}^{\infty}\left(\frac{2r-1}{r}\right)$ diverges.

8. Use integral test to show that the harmonic series $\sum_{r=1}^{\infty}\frac{1}{r}$ is divergent.

9. Using d'Alembert ratio test, test the convergence or otherwise of the following series

   (a) $\sum_{r=1}^{\infty}\frac{5^r}{r}$

   (b) $\sum_{r=1}^{\infty}\frac{a^r}{r^2}, \quad a > 0$

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5 Module Five: FUNCTIONS

5.1 Introduction

In this topic, we shall discuss continuity of functions and their properties. We shall also discuss differentiability of functions as well as Taylor's theorem.

Objectives

At the end of this module, students should be able to:

a. define and recognize the real functions and their limits;

b. prove various theorems about limits of functions and emphasize the proofs' development;

c. define continuity of a function;

d. prove various theorems about continuous functions;

e. define the derivative of a function;

f. prove various theorems about the derivatives of functions.

Pre-Test

a. Evaluate

(i) $\lim_{x \to 1} \frac{x^3 - 1}{x - 1}$

(ii) $\lim_{x \to 0} \frac{\sin x}{x}$

b. What is a continuous function?

c. Differentiate each of the following functions with respect to x

(i) $\frac{x + 1}{x - 1}$

(ii) $e^x \sin x$

5.2 Limits of Functions

From the definition of limit of a sequence, $\lim_{x \to \infty} x_n = x_0$ means that x_n is arbitrarily close to x_0 if n is sufficiently large. The definition of a function limit is intended in much the same way. That is

$$\lim_{x \to a} f(x) = L$$

means that $f(x)$ is arbitrarily close to L if x is sufficiently close to a.

It is important to note from the above statement that $f(a)$ is not part of the consideration. It is possible that $f(a) = L$, but whether this is true or false should not be any influence on the existence of the limit.

Definition 5.1. For a real-valued function f, we say that

$$\lim_{x\to a} f(x) = L$$

if and only if the following conditions are satisfied

(a) f is defined on some interval, Γ_a, around a, where Γ_a is defined by

$$\Gamma_a = \{x : 0 < |x - a| < \delta\}$$

(b) for every sequence $\{x_n\}$ in Γ_a such that $x_n \to a$, we have

$$\lim_{n\to\infty} f(x_n) = L.$$

5.2.1 One-Sided Limit

It is possible for a function to fail to have a limit at a point and yet appear to have limits on one side. So we introduce the following definition

Definition 5.2. (a) (Limit from the right) $\lim\limits_{x\to a^+} f(x) = c$ if the following two conditions are satisfied

(i) f is defined on some interval $(a, a+\delta)$, $\delta > 0$;

(ii) if $\{x_n\}$ is an arbitrary sequence in $(a, a+\delta)$ such that $x_n \to a$, then $f(x_n) \to c$ as $n \to \infty$.

(b) $\lim\limits_{x\to a^-} f(x) = c$ is defined similarly.

Proposition 5.3. *Let $f : D(f) \subset \mathbb{R} \to \mathbb{R}$ be any map. Then*

$$\lim_{x\to a} f(x) = L \qquad \textit{iff} \qquad \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L$$

Proof. Suppose $\lim\limits_{x\to a} f(x) = L$. Therefore f is defined on □

Theorem 5.4. *Suppose $f(x) \to L_1$ as $x \to a$ and $g(x) \to L_2$ as $x \to a$, then*

(i) $f(x) + g(x) \to L_1 + L_2$ as $x \to a$;

(ii) $f(x) - g(x) \to L_1 - L_2$ as $x \to a$;

(iii) $f(x)g(x) \to L_1L_2$ as $x \to a$;

(iv) $\dfrac{f(x)}{g(x)} \to \dfrac{L_1}{L_2}$ provided $g(x) \neq 0\ \forall\ x$, as $x \to a$

Can a function have two different limits?

Theorem 5.5. *(Uniqueness of Limits) Suppose that*

$$\lim_{x\to x_0} f(x) = L\ ,$$

then the number L is unique.

Proof. It follows from the uniqueness of limit of a sequence □

5.3 Continuity of Functions

Definition 5.6. Let $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$ be a map with domain $D(f) \subseteq \mathbb{R}$. Then f is said to be continuous at $x_0 \in D(f)$ if given any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \qquad \text{whenever} \qquad |x - x_0| < \delta$$

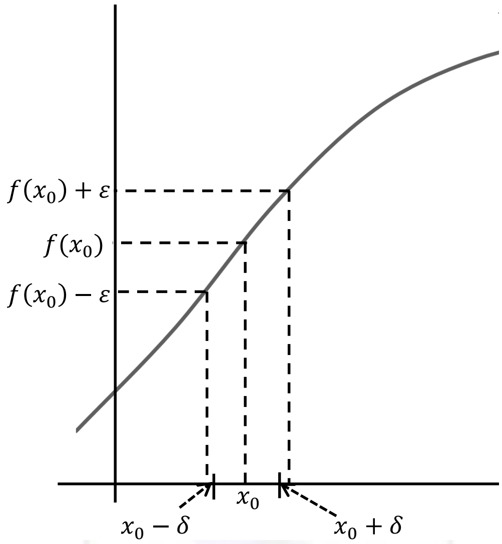

Figure 5.1: Continuity at x_0

Example 5.7. Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 2x^2 - 1$ is continuous at $x = 2$.

Solution

Let

We now give another definition of continuity which is very convenient in applications

Definition 5.8. A function $f : D(f) \subseteq \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in D(f)$ if for any sequence $\{x_n\}$ in $D(f)$ such that $x_n \to x_0$, then $f(x_n) \to f(x_0)$.

Theorem 5.9. *Definitions 5.6 and 5.8 are equivalent*

In the sequel, we shall be using Definition 5.8 to establish important results on continuity of functions.

Theorem 5.10. *Let $f, g : J \to \mathbb{R}$ be continuous at $a \in J$ and let $\alpha \in \mathbb{R}$, then*

(i) $f + g$ is continuous at a;

(ii) $f - g$ is continuous at a;

(iii) αf is continuous at a;

(iv) fg is continuous at a;

Proof. (i) Let $\{x_n\}$ be a sequence in J such that $x_n \to a$. We need to prove that $(f+g)(x_n) \to (f+g)(a)$. By definition, $(f+g)(x_n) = f(x_n) + g(x_n)$. Since f and g are continuous at a, we have $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$. By the algebra of convergent sequences, we have $f(x_n) + g(x_n) \to f(a) + g(a)$. This means $(f+g)(x_n) \to (f+g)(a)$.

(ii) Similar to the prove of (i) and left as an exercise for the student

(iii) Similar to the prove of (i) and left as an exercise for the student

(iv) Let $\{x_n\}$ be a sequence in J such that $x_n \to a$. We need to prove that $(fg)(x_n) \to (fg)(a)$. Now, $(fg)(x_n) = f(x_n)g(x_n)$. Since f and g are continuous at a, we have $f(x_n) \to f(a)$ and $g(x_n) \to g(a)$. By the algebra of convergent sequences, we have $f(x_n)g(x_n) \to f(a)g(a)$. This means $(fg)(x_n) \to (fg)(a)$. □

We then state the most important global results on continuity.

Theorem 5.11. *(Intermediate Value Theorem) Let $f[a,b] \to \mathbb{R}$ be a continuous function. Let λ be a real number between $f(a)$ and $f(b)$. Then there exists $c \in (a,b)$ such that $f(c) = \lambda$.*

Remark 5.12. We assume in Theorem 5.11 that the domain is an interval. If the domain is not an interval, the conclusion does not remain valid.

Theorem 5.13. *(Weierstrass Theorem) Let $f : [a,b] \to \mathbb{R}$ be a continuous function. Then f is bounded.*

This theorem explains that every continuous function on $[a,b]$ is bounded on $[a,b]$. However, the converse is not true in general.

Remark 5.14. If the domain is not bounded or if the domain is not closed, then Weierstrass theorem is not true. For example, $f(x) = \dfrac{1}{x}$ is not bounded on $(0,1)$ and $g(x) = x$ is not bounded on $(0,\infty)$.

5.4 Differentiability

Definition 5.15. Let J be an interval and $c \in J$. Let $f : J \to \mathbb{R}$. Then f is said to be differentiable at c if there exists a real number α such that

$$\lim_{x\to c} \frac{f(x) - f(c)}{x - c} = \alpha \tag{5.1}$$

If we take $h = x - c$, (5.1) can be written as

$$\lim_{h\to 0} \frac{f(c+h) - f(c)}{h} = \alpha \tag{5.2}$$

(5.1) can also be defined in $\varepsilon - \delta$ form.

Definition 5.16. We say that the derivative of f at c is $\alpha \in \mathbb{R}$ if for any given $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$x \in J \quad \text{and} \quad 0 < |x - c| < \delta \quad \Rightarrow \quad |f(x) - f(c) - \alpha(x - c)| < \varepsilon|x - c| \tag{5.3}$$

We say that f is differentiable on J if it is differentiable at each $c \in J$.

Example 5.17. Let $f : J \to \mathbb{R}$ be given by $f(x) = ax + b$. Show that $f'(x) = a$.

Solution

Let $\varepsilon > 0$ be given, then

$$|f(x) - f(c) - \alpha(x - c)| = |ax + b - (ac + b) - a(x - c)| = 0$$

This suggests that we can choose any $\delta > 0$ for the given $\varepsilon > 0$. Since c was arbitrary, f is differentiable on $\mathbb{R}$ and $f'(x) = a$.

Example 5.18. If $f : J \to \mathbb{R}$ is given by $f(x) = 4x^2$, show that f is differentiable on $\mathbb{R}$ and $f'(x) = 8x$.

Solution

$$\begin{aligned} |f(x) - f(c) - \alpha(x - c)| &= |4x^2 - 4c^2 - 8c(x - c)| \\ &= |4(x - c)(x + c) - 8c(x - c)| \\ &= 4|x - c||x - c| \end{aligned}$$

Now, given $\varepsilon > 0$, choose $\delta = \min\left\{1, \frac{\varepsilon}{4}\right\}$. For x such that $0 < |x - c| < \delta$, we have

$$|f(x) - f(c) - \alpha(x - c)| < 4\delta|x - c| < \varepsilon|x - c|.$$

Since $c \in \mathbb{R}$ was arbitrary, f is differentiable on $\mathbb{R}$ and $f'(x) = 4x$.

We now prove an important result

Theorem 5.19. *If f is differentiable at a, then f is continuous at a.*

Proof. We use a less rigorous approach. Let $a \in D(f)$ be arbitrary. We need to show that f is continuous at a. Now, for any $x \in D(f)$, $x \neq a$, we have

$$f(x) = (x - a)\frac{f(x) - f(a)}{x - a} + f(a) \tag{5.4}$$

Since f is differentiable at $x = a$, $f'(a)$ exists ie

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = f'(a)$$

Now from (5.4),

$$\begin{aligned} \lim_{x \to a} f(x) &= \lim_{x \to a}(x - a) \lim_{x \to a} \frac{f(x) - f(a)}{x - a} + \lim_{x \to a} f(a) \\ &= 0 \cdot f'(a) + f(a) \\ &= f(a). \end{aligned}$$

□

Theorem 5.20. *Let f and g be both differentiable at a. Then*

(i) if $h(x) = f(x) + g(x)$, then $h'(a) = f'(a) + g'(a)$

(ii) if $h(x) = f(x)g(x)$, then $h'(a) = f(a)g'(a) + g(a)f'(a)$

(iii) if $h(x) = \dfrac{f(x)}{g(x)}$ and $g(a) \neq 0$, then $h'(a) = \dfrac{g(a)f'(a) - f(a)g'(a)}{[g(a)]^2}$

Proof. (i) If $h(x) = f(x) + g(x)$, then

$$\begin{aligned} h'(a) &= \lim_{x\to a} \frac{h(x) - h(a)}{x - a} \\ &= \lim_{x\to a} \frac{f(x) + g(x) - f(a) - g(a)}{x - a} \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} + \lim_{x\to a} \frac{g(x) - g(a)}{x - a} \\ &= f'(a) + g'(a) \end{aligned}$$

Similarly, if $h(x) = f(x) - g(x)$, then $h'(x) = f'(x) - g'(x)$.

(ii) If $h(x) = f(x)g(x)$, then

$$\begin{aligned} h'(a) &= \lim_{x\to a} \frac{f(x)g(x) - f(a)g(a)}{x - a} \\ &= \lim_{x\to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a))}{x - a} \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x\to a} g(x) + \lim_{x\to a} \frac{g(x) - g(a)}{x - a} f(a) \\ &= f'(a)g(a) + g'(a)f(a) \end{aligned}$$

(iii) If $h(x) = \dfrac{f(x)}{g(x)}$ and $g(a) \neq 0$, then

$$\begin{aligned} h'(a) &= \lim_{x\to a} \frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x - a} \\ &= \lim_{x\to a} \frac{f(x)g(a) - f(a)g(x)}{(x - a)g(x)g(a)} \\ &= \lim_{x\to a} \frac{f(x)g(x) - f(a)g(x) - f(x)g(x) + f(x)g(a)}{(x - a)g(x)g(a)} \\ &= \lim_{x\to a} \frac{f(x) - f(a)}{(x - a)g(a)} - \lim_{x\to a} \frac{g(x) - g(a)}{(x - a)g(a)} \cdot \lim_{x\to a} \frac{f(x)}{g(x)} \\ &= \frac{f'(a)}{g(a)} - \frac{f(a)g'(a)}{[g(a)]^2} \\ &= \frac{f'(a)g(a - f(a)g'(a))}{[g(a)]^2} \end{aligned}$$

□

We now give the chain rule

Theorem 5.21. *(Chain rule) Let $f : J \to \mathbb{R}$ be differentiable and $f(J) \subset J_1$, an interval and if $g : J_1 \to \mathbb{R}$ is differentiable at $f(c)$, then $g\circ f$ is differentiable at c with $(g\circ f)'(c) = g'(f(c))\cdot f'(c)$.*

Definition 5.22. Let $J \subset \mathbb{R}$ be an interval and $f : J \to \mathbb{R}$ be a function. We say that a point $c \in J$ is a local maximum point if there exists $\delta > 0$ such that $(c-\delta, c+\delta) \subset J$ and $f(x) \leq f(c)$ for all $x \in (c-\delta, c+\delta)$. A local minimum point is defined similarly. A point c is said to be a local extremum point if it is either a local maximum point or a local minimum point.

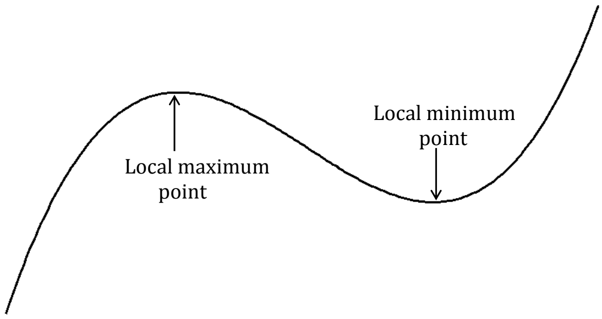

Figure 5.2: Local maximum and minimum points

Following are the main theorems of differentiability of function.

Theorem 5.23. *(Rolle's theorem) Suppose*

(i) f *is continuous on* $[\alpha, \beta]$;

(ii) $f(\alpha) = f(\beta)$;

(iii) f' *exists on* (α, β),

then f *has a local maximum or minimum at some* $c \in (\alpha, \beta)$ *and that* $f'(c) = 0$.

Theorem 5.24. *(Mean Value Theorem) Let* $f : [a, b] \to \mathbb{R}$ *be such that*

(i) f *is continuous on* $[a, b]$;

(ii) f *is differentiable on* (a, b).

Then there exists $c \in (a, b)$ *such that*

$$f(b) - f(a) = f'(c)(b - a)$$

Proof. Consider

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Clearly $g(a) = g(b) = 0$. Therefore g satisfies the conditions of Rolle's theorem (Theorem 5.23). Hence there exists $c \in (a, b)$ such that $g'(c) = 0$. This implies

$$f'(c) - \frac{f(b) - f(a)}{b - a} = 0.$$

□

Note that the Mean Value Theorem (MVT) does not tell us what c is. It only tells us that there is at least one number c that will satisfy the conclusion of the theorem.
The MVT has applications in the theory of differential equations and numerical analysis.

Post Test

1. Give a counter example to show that the converse of Theorem 5.19 is not true.
2. If $f : J \to \mathbb{R}$ is given by $f(x) = x^3$, show that f is differentiable on $\mathbb{R}$ and $f'(x) = 3x^2$.
3. If $f : J \to \mathbb{R}$ is given by $f(x) = \dfrac{1}{x^2+1}$, show that f is differentiable on $\mathbb{R}$ and $f'(x) = -\dfrac{2x}{(x^2+1)^2}$.
4. Let $f, g : J \to \mathbb{R}$ be continuous at $a \in J$ and let $\alpha, \beta \in \mathbb{R}$, then
   (i) $f - g$ is continuous at a;
   (ii) αf is continuous at a;
   (iii) $\alpha f + \beta g$ is continuous at a;
5. If $\lim_{x\to x_0} f(x) = L$, $\lim_{x\to x_0} g(x) = M$ and $\alpha, \beta \in \mathbb{R}$, prove that $\lim_{x\to x_0} (\alpha f(x) - \beta g(x)) = \alpha L - \beta M$.
6. Prove that $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \dfrac{2x}{x^2+1}$ is continuous at $x = 1$.

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