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Viscosity iterative techniques for approximating a common zero of monotone operators in an Hadamard space

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Abstract
The main purpose of this paper is to introduce some viscosity-type proximal point algorithms which comprise of a nonexpansive mapping and a finite sum of resolvents of monotone operators, and prove their strong convergence to a common zero of a finite family of monotone operators which is also a fixed point of a nonexpansive mapping and a unique solution of some variational inequality problems in an Hadamard space. We apply our results to solve a finite family of convex minimization problems, variational inequality problems and convex feasibility problems.

Keywords Monotone operators · Convex feasibility problems · Variational inequalities · Minimization problems · Viscosity iterations · CAT(0) space

Mathematics Subject Classification 47H09 · 47H10 · 49J20 · 49J40

1 Introduction

Let X be a complete metric space, then X is called an Hadamard space if it is geodesically connected, and if every geodesic triangle in X is at least as thin as its comparison triangle in the Euclidean plane. The extension of known concepts from Hilbert, Banach and topological

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vector spaces, as well as Hadamard manifolds to Hadamard spaces has been of great interest to many researchers in this direction. One of such known concept is the theory of monotone operators which is known to be one of the most important notions in optimization theory. Monotone operator theory is an area of research in mathematics that has received a lot of attention, and has enjoyed many prosperous developments in recent time. An important problem in monotone operator theory is the problem of finding a zero of a monotone operator, defined as: find $x \in D(A)$ such that

$$0 \in Ax, \tag{1.1}$$

where A is a monotone mapping (to be defined in Sect. 2) and $D(A)$ denotes the domain of A. The solution set of problem (1.1) is known to be closed and convex (see [39]) and we denote it by $A^{-1}(0)$. Many optimization problems can be modelled as problem (1.1). For instance, the problem of finding a solution of a minimization problem for a proper convex and lower semicontinuous function (see for example, [2,15,16,21,23,24,38,45]) can be modelled as problem (1.1). In this case, the monotone operator is the subdifferential of the convex functional. Also, a solution of problem (1.1) is a solution of a variational inequality problem (we shall discuss these in details in Sect. 4). Moreover, the problem of finding a zero of monotone operators describes the equilibrium or stable state of an evolution system governed by the monotone mapping, which is very important in ecology, physics, economics, among others (see [5,6,13,22,25,28,34,39,49] and the references therein). Thus, researchers in this area have devoted a lot of efforts in developing different iterative techniques for approximating solutions of problem (1.1). A classical iterative technique for approximating solutions of problem (1.1) is the Proximal Point Algorithm (PPA), which was introduced in Hilbert spaces by Martinet [33] and was further developed by Rockafellar [40] for approximating solutions of (1.1) in a real Hilbert space H as follows: for arbitrary $x_0 \in H$, the sequence $\{x_n\}$ is generated by

$$x_{n-1} - x_n \in \lambda_n A(x_n), \tag{1.2}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers. Rockafellar [40] proved that the sequence $\{x_n\}$ generated by Algorithm (1.2) is weakly convergent to a solution of (1.1), provided $\lambda_n \geq \lambda > 0$ for each $n \geq 1$. Since then, many other researchers have developed and studied different modifications of the PPA for finding solutions of (1.1) in both Hilbert and Banach spaces (see [9,11,14,27,35–37,43]), as well as Hadamard manifolds (see [31,48] and the references therein).

The study of the PPA for approximating solutions of problem (1.1) has recently been extended from these spaces to Hadamard spaces. For instance, Khatibzadeh and Ranjbar [28] introduced and studied the following PPA in an Hadamard space for approximating a zero of a monotone operator, for which they obtained a Δ-convergence result:

$$\begin{cases} x_0 \in X, \\ x_n = J_{\lambda_n}^A x_{n-1}, \end{cases} \tag{1.3}$$

where $J_{\lambda_n}^A$ is the resolvent of the monotone mapping A (to be defined in Sect. 2) with sequence $\{\lambda_n\} \subset (0, \infty)$ such that $\sum_{n=1}^{\infty} \lambda_n = \infty$. They also obtained a strong convergence result using the above PPA under the assumption that A is strongly monotone.

Very recently, Ranjbar and Khatibzadeh [39] proposed and studied the following Mann-type and Halpern-type PPA in Hadamard spaces for approximating solutions of (1.1):

$$\begin{cases} x_0 \in X, \\ x_{n+1} = \alpha_n x_n \oplus (1 - \alpha_n) J^A_{\lambda_n} x_n \end{cases} \tag{1.4}$$

and

$$\begin{cases} u, x_0 \in X, \\ x_{n+1} = \alpha_n u \oplus (1 - \alpha_n) J^A_{\lambda_n} x_n, \end{cases} \tag{1.5}$$

where $\{\lambda_n\} \subset (0, \infty)$ and $\{\alpha_n\} \subset [0, 1]$. Using (1.4) and (1.5), they obtained Δ-convergence and strong convergence results respectively, under some suitable conditions.

Motivated by the results of Khatibzadeh and Ranjbar [28], and the results of Ranjbar and Khatibzadeh [39], we introduce some viscosity-type PPAs which comprise of a nonexpansive mapping and a finite sum of resolvents of monotone operators, and prove their strong convergence to a common zero of a finite family of monotone operators which is also a fixed point of a nonexpansive mapping and a unique solution of some variational inequality problems in an Hadamard space. Furthermore, we apply our results to solve a finite family of convex minimization problems, variational inequality problems and convex feasibility problems. Our results extend and improve the results of Khatibzadeh and Ranjbar [28], Ranjbar and Khatibzadeh [39] and many other important results in this direction.

2 Preliminaries

In this section, we recall some basic and useful results that will be needed in establishing our main results.

Definition 2.1 Let (X, d) be a metric space, $x, y \in X$ and $I = [0, d(x, y)]$ be an interval. A curve c (or simply a geodesic path) joining x to y is an isometry $c : I \to X$ such that $c(0) = x, c(d(x, y)) = y$ and $d(c(t), c(t') = |t - t'|$ for all $t, t' \in I$. The image of a geodesic path is called the geodesic segment, which is denoted by $[x, y]$ whenever it is unique.

Definition 2.2 [19] A metric space (X, d) is called a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if every two points of X are joined by exactly one geodesic. A subset C of X is said to be convex if C includes every geodesic segments joining two of its points. Let $x, y \in X$ and $t \in [0, 1]$, we write $tx \oplus (1 - t)y$ for the unique point z in the geodesic segment joining from x to y such that

$$d(x, z) = (1 - t)d(x, y) \quad \text{and} \quad d(z, y) = td(x, y). \tag{2.1}$$

A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three vertices (points in X) with unparameterized geodesic segment between each pair of vertices. For any geodesic triangle, there is comparison (Alexandrov) triangle $\bar{\Delta} \subset \mathbb{R}^2$ such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$ for $i, j \in \{1, 2, 3\}$. Let Δ be a geodesic triangle in X and $\bar{\Delta}$ be a comparison triangle for $\bar{\Delta}$, then Δ is said to satisfy the CAT(0) inequality if for all points $x, y \in \Delta$ and $\bar{x}, \bar{y} \in \bar{\Delta}$,

$$d(x, y) \leq d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{2.2}$$

Let x, y, z be points in X and y_0 be the midpoint of the segment $[y, z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d(y, z). \tag{2.3}$$

Inequality (2.3) is known as the CN inequality of Bruhat and Titis [10].

Definition 2.3 A geodesic space X is said to be a CAT(0) space if all geodesic triangles satisfy the CAT(0) inequality. Equivalently, X is called a CAT(0) space if and only if it satisfies the CN inequality.

CAT(0) spaces are examples of uniquely geodesic spaces and complete CAT(0) spaces are called Hadamard spaces.

Definition 2.4 [7] Let X be a CAT(0) space. Denote the pair $(a, b) \in X \times X$ by $\overrightarrow{ab}$ and call it a vector. Then, a mapping $\langle ., . \rangle : (X \times X) \times (X \times X) \to \mathbb{R}$ defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2} \left(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d) \right) \quad \forall a, b, c, d \in X$$

is called a quasilinearization mapping.

It is easily to check that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$ for all $a, b, c, d, e \in X$. A geodesic space X is said to satisfy the Cauchy–Schwarz inequality if $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d)$ $\forall a, b, c, d \in X$. It has been established in [7] that a geodesically connected metric space is a CAT(0) space if and only if it satisfies the Cauchy–Schwarz inequality. Examples of CAT(0) spaces includes: Euclidean spaces $\mathbb{R}^n$, Hilbert spaces, simply connected Riemannian manifolds of nonpositive sectional curvature, $\mathbb{R}$-trees, Hilbert ball [20], Hyperbolic spaces [44], among others.

Based on the notion of quasilinearization mapping, the notion of the dual space of an Hadamard space X was introduced by Kakavandi and Amini [1] by first introducing the concept of pseudometric space which they defined as the space $\mathbb{R} \times X \times X$ endowed with a pseudometric D, defined as

$$D((t, a, b), (s, c, d)) = L(\Theta(t, a, b) - \Theta(s, c, d)), \ (t, s \in \mathbb{R}, \ a, b, c, d \in X),$$

where $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ is defined as $\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle$, for all $t \in \mathbb{R}$, $a, b, x \in X$, ($C(X, \mathbb{R})$ is the space of all continuous real-valued functions on X) and $L(\varphi) = \sup \left\{ \frac{\varphi(x) - \varphi(y)}{d(x, y)} : x, y \in X, \ x \neq y \right\}$ for any function $\varphi : X \to \mathbb{R}$. Moreover, D defines an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t, a, b) is $[t\overrightarrow{ab}] := \{ s\overrightarrow{cd} : D((t, a, b), (s, c, d)) = 0 \}$. Then, the dual space of (X, d) is the metric space $X^* = \{ [t\overrightarrow{ab}] : (t, a, b) \in \mathbb{R} \times X \times X \}$ together with the metric D. Furthermore, the dual space X^* acts on $X \times X$ by $\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle$, $x^* = [t\overrightarrow{ab}] \in X^*$, $x, y \in X$ (see [28]).

Definition 2.5 Let X be an Hadamard space. A point $x \in X$ is called a fixed point of a nonlinear mapping $T : X \to X$, if $Tx = x$. The set of fixed points of T is denoted by $F(T)$. The mapping T is said to be

(i) *firmly nonexpansive* (see [28]) if

$$d^2(Tx, Ty) \leq \langle \overrightarrow{TxTy}, \overrightarrow{xy} \rangle \quad \forall x, y \in X,$$

(ii) *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y) \quad \forall x, y \in X.$$

From Cauchy–Schwarz inequality, it is clear that the class of nonexpansive mappings is more general than the class of firmly nonexpansive mappings.

Definition 2.6 [28]. Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be a multivalued operator such that $D(A) := \{x \in X : Ax \neq \emptyset\}$. Then, A is called monotone if

$$\langle x^* - y^*, \overrightarrow{yx} \rangle \geq 0 \quad \forall x, y \in D(A),\ x^* \in Ax,\ y^* \in Ay.$$

Definition 2.7 [28] Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be any multivalued operator. Then, the resolvent of A of order $\lambda > 0$ is a mapping $J_\lambda^A : X \to X$ defined by

$$J_\lambda^A(x) := \left\{ z \in X \mid \left[\frac{1}{\lambda} \overrightarrow{zx} \right] \in Az \right\}. \tag{2.4}$$

We say that a monotone operator A satisfies the range condition if for every $\lambda > 0,\ D(J_\lambda^A) = X$ (see [28]).

The relationship between monotone operators and their resolvents in a CAT(0) space is given in the following result which plays vital roles in establishing our main results.

Theorem 2.8 [28] *Let X be a CAT(0) space and J_λ^A be the resolvent of a multivalued mapping A of order λ. Then,*

(i) *for any $\lambda > 0,\ R(J_\lambda^A) \subset D(A)$ and $F(J_\lambda^A) = A^{-1}(0)$, where $R(J_\lambda^A)$ is the range of J_λ^A,*
(ii) *if A is monotone then J_λ^A is a single-valued and firmly nonexpansive mapping,*
(iii) *if A is monotone and $0 < \lambda \leq \mu$, then $d^2(J_\lambda^A x, J_\mu^A x) \leq \frac{\mu-\lambda}{\mu+\lambda} d^2(x, J_\mu^A x)$.*

Definition 2.9 Let $\{x_n\}$ be a bounded sequence in a geodesic metric space X. Then, the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined by

$$A(\{x_n\}) = \left\{ \bar{v} \in X : \limsup_{n\to\infty} d(\bar{v}, x_n) = \inf_{v \in X} \limsup_{n\to\infty} d(v, x_n) \right\}.$$

It is generally known that in an Hadamard space, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $\bar{v} \in X$ if $A(\{x_{n_k}\}) = \{\bar{v}\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n\to\infty} x_n = \bar{v}$ (see [18]). The concept of Δ-convergence in metric spaces was first introduced and studied by Lim [32]. Kirk and Panyanak [30] later introduced and studied this concept in CAT(0) spaces, and proved that it is very similar to the weak convergence in Banach space setting. In this connection, see also [42].

The following lemmas will be very useful in proving our main results.

Lemma 2.10 *Let X be a CAT(0) space, $x, y, z \in X$ and $t, s \in [0, 1]$. Then*

(i) $d(tx \oplus (1-t)y, z) \leq td(x, z) + (1-t)d(y, z)$ (see [19]).
(ii) $d^2(tx \oplus (1-t)y, z) \leq td^2(x, z) + (1-t)d^2(y, z) - t(1-t)d^2(x, y)$ (see [19]).
(iii) $d^2(tx \oplus (1-t)y, z) \leq t^2 d^2(x, z) + (1-t)^2 d^2(y, z) + 2t(1-t)\langle \overrightarrow{xz}, \overrightarrow{yz} \rangle$ (see [17]).
(iv) $d(tw \oplus (1-t)x, ty \oplus (1-t)z) \leq td(w, y) + (1-t)d(x, z)$ (see [8]).
(v) $z = tx \oplus (1-t)y$ implies $\langle \overrightarrow{zy}, \overrightarrow{zw} \rangle \leq t\langle \overrightarrow{xy}, \overrightarrow{zx} \rangle, \quad \forall\ w \in X$ (see [17]).
(vi) $d(tx \oplus (1-t)y, sx \oplus (1-s)y) \leq |t-s| d(x, y)$ (see [12]).

Lemma 2.11 [50] *Let X be a CAT(0) space. For any $t \in [0, 1]$ and $u, v \in X$, let $u_t = tu \oplus (1-t)v$. Then, for all $x, y \in X$,*

(1) $\langle\overrightarrow{u_t x}, \overrightarrow{u_t y}\rangle \le t\langle\overrightarrow{ux}, \overrightarrow{u_t y}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{u_t y}\rangle$;
(2) $\langle\overrightarrow{u_t x}, \overrightarrow{uy}\rangle \le t\langle\overrightarrow{ux}, \overrightarrow{uy}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{ux}\rangle$ *and*
(3) $\langle\overrightarrow{u_t x}, \overrightarrow{vy}\rangle \le t\langle\overrightarrow{ux}, \overrightarrow{vy}\rangle + (1-t)\langle\overrightarrow{vx}, \overrightarrow{vy}\rangle$.

Lemma 2.12 [19] *Every bounded sequence in an Hadamard space always have a Δ-convergent subsequence.*

Lemma 2.13 ([41, Opial's lemma]) *Let X be an Hadamard space and $\{x_n\}$ be a sequence in X. If there exists a nonempty subset F in which*

(i) $\lim_{n\to\infty} d(x_n, z)$ *exists for every $z \in F$, and*
(ii) *if $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ which is Δ-convergent to x, then $x \in F$.*

Then, there is a $p \in F$ such that $\{x_n\}$ is Δ-convergent to p in X.

Lemma 2.14 [26] *Let X be an Hadamard space, $\{x_n\}$ be a sequence in X and $x \in X$. Then, $\{x_n\}\Delta$-converges to x if and only if* $\limsup_{n\to\infty}\langle\overrightarrow{xx_n}, \overrightarrow{xy}\rangle \le 0$ *for all $y \in C$.*

Lemma 2.15 [46] *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a metric space of hyperbolic type X and $\{\beta_n\}$ be a sequence in $[0, 1]$ with* $\liminf_{n\to\infty}\beta_n < \limsup_{n\to\infty}\beta_n < 1$. *Suppose that $x_{n+1} = \beta_n x_n \oplus (1-\beta_n)y_n$ for all $n \ge 0$ and* $\limsup_{n\to\infty}(d(y_{n+1}, y_n) - d(x_{n+1}, x_n)) \le 0$. *Then* $\lim_{n\to\infty} d(y_n, x_n) = 0$.

Lemma 2.16 [51] *Let $\{a_n\}$ be a sequence of nonnegative real numbers satisfying the following relation:*

$$a_{n+1} \le (1-\alpha_n)a_n + \alpha_n\sigma_n + \gamma_n, \quad n \ge 0,$$

where (i) $\{\alpha_n\} \subset [0, 1]$, $\sum\alpha_n = \infty$; (ii) $\lim sup\, \sigma_n \le 0$; (iii) $\gamma_n \ge 0$; $(n \ge 0)$, $\sum\gamma_n < \infty$. Then, $a_n \to 0$ as $n \to \infty$.

Remark 2.17 (See also [47]). For a CAT(0) space X, if $\{x_i, i = 1, 2, \ldots, N\} \subset X$, and $\alpha_i \in [0, 1]$, $i = 1, 2, \ldots, N$. Then by induction, we can write

$$\bigoplus_{i=1}^{N}\alpha_i x_i := (1-\alpha_N)\bigoplus_{i=1}^{N-1}\frac{\alpha_i}{1-\alpha_N}x_i \oplus \alpha_N x_N. \tag{2.5}$$

3 Main results

Lemma 3.1 *Let X be a CAT(0) space, $\{x_i,\ i = 1, 2, \ldots, N\} \subset X$, $\{y_i,\ i = 1, 2, \ldots, N\} \subset X$ and $\alpha_i \in [0, 1]$ for each $i = 1, 2, \ldots, N$ such that $\sum_{i=1}^{N}\alpha_i = 1$. Then,*

$$d\left(\bigoplus_{i=1}^{N}\alpha_i x_i, \bigoplus_{i=1}^{N}\alpha_i y_i\right) \le \sum_{i=1}^{N}\alpha_i d(x_i, y_i). \tag{3.1}$$

Proof (By induction). For $N = 2$, the result follows from Lemma 2.10 (iv). Now, assume that (3.1) holds for $N = k$, for some $k \ge 2$. Then, we prove that (3.1) also holds for $N = k + 1$. Indeed, by Remark 2.17, Lemma 2.10 (iv) and our assumption, we obtain that

$$d\left(\bigoplus_{i=1}^{k+1}\alpha_i x_i, \bigoplus_{i=1}^{k+1}\alpha_i y_i\right) \leq (1-\alpha_{k+1})d\left(\bigoplus_{i=1}^{k}\frac{\alpha_i}{1-\alpha_{k+1}}x_i, \bigoplus_{i=1}^{k}\frac{\alpha_i}{1-\alpha_{k+1}}y_i\right)$$
$$+\alpha_{k+1}d(x_{k+1}, y_{k+1})$$
$$\leq \sum_{i=1}^{k+1}\alpha_i d(x_i, y_i).$$

Hence, (3.1) holds for all $N \in \mathbb{N}$. □

Lemma 3.2 *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be a finite family of multivalued monotone operators and $T : X \to X$ be a nonexpansive mapping. Then, for $\beta_i \in (0, 1)$ with $\sum_{i=0}^{N}\beta_i = 1$, the mapping S_λ defined by $S_\lambda x := \beta_0 x \oplus \beta_1 J_\lambda^{A_1}x \oplus \beta_2 J_\lambda^{A_2}x \oplus \cdots \oplus \beta_N J_\lambda^{A_N}x$ is nonexpansive and $F(T \circ S_\mu) \subseteq \bigcap_{i=1}^{N} F(J_\lambda^{A_i}) \cap F(T)$ for all $x \in X$, $0 < \lambda \leq \mu$.*

Proof Since A_i is monotone for each $i = 1, 2, \ldots, N$, it follows from Theorem 2.8 that $J_\lambda^{A_i}$ is single-valued and nonexpansive for $\lambda > 0$, $i = 1, 2, \ldots, N$. Thus, by Lemma 3.1, we obtain

$$d(S_\lambda x, S_\lambda y) \leq \beta_0 d(x, y) + \beta_1 d(J_\lambda^{A_1}x, J_\lambda^{A_1}y) + \cdots + \beta_N d(J_\lambda^{A_N}x, J_\lambda^{A_N}y)$$
$$\leq \sum_{i=0}^{N}\beta_i d(x, y)$$
$$= d(x, y).$$

Hence, S_λ is nonexpansive.

Now, let $x \in F(T \circ S_\mu)$ and $v \in \bigcap_{i=1}^{N} F(J_\mu^{A_i}) \cap F(T)$. Then, by Lemma 3.1, we obtain

$$\begin{aligned} d(x, v) &\leq d(S_\mu x, v) \\ &\leq \beta_0 d(x, v) + \beta_1 d(J_\mu^{A_1}x, v) + \cdots + \beta_N d(J_\mu^{A_N}x, v) \\ &\leq \sum_{i=0}^{N-1}\beta_i d(x, v) + \beta_N d(J_\mu^{A_N}x, v) \\ &\leq d(x, v). \end{aligned} \tag{3.2}$$

From (3.2), we obtain that

$$d(x, v) = \sum_{i=0}^{N-1}\beta_i d(x, v) + \beta_N d(J_\mu^{A_N}x, v) = (1-\beta_N)d(x, v) + \beta_N d(J_\mu^{A_N}x, v),$$

which implies that $d(x, v) = d(J_\mu^{A_N}x, v)$. Similarly, we obtain

$$d(x, v) = d(J_\mu^{A_{N-1}}x, v) = \cdots = d(J_\mu^{A_2}x, v) = d(J_\mu^{A_1}x, v).$$

By the uniform convexity of X, we obtain that

$$x = J_\mu^{A_i}x, \quad i = 1, 2, \ldots, N. \tag{3.3}$$

Thus, $d(x, Tx) = d(TS_\mu x, Tx) \leq d(S_\mu x, x) \leq 0$, which implies that $x = Tx$.

Since $0 < \lambda \leq \mu$, we obtain from Theorem 2.8 (iii) and (3.3) that

$$d(x, J_\lambda^{A_i}x) \leq 2d(x, J_\mu^{A_i}x) = 0, \ i = 1, 2, \ldots, N.$$

Hence, $x = J_\lambda^{A_i}x$, $i = 1, 2, \ldots, N$. Therefore, we conclude that $F(T \circ S_\mu) \subseteq \bigcap_{i=1}^N F(J_\lambda^{A_i}) \cap F(T)$. □

We now present our strong convergence theorems.

Theorem 3.3 *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be a finite family of multivalued monotone operators that satisfy the range condition. Let T be a nonexpansive mapping on X and h be a contraction mapping on X with coefficient $\tau \in (0, 1)$. Suppose that $\Gamma := F(T) \bigcap \left(\cap_{i=1}^N A_i^{-1}(0)\right) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \beta_2 J_{\lambda_n}^{A_2} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n, \\ x_n = \alpha_n h(x_n) \oplus (1 - \alpha_n) T y_n, \quad n \geq 1, \end{cases} \tag{3.4}$$

where $0 < \lambda \leq \lambda_n$ $\forall n \geq 1$ and $\{\alpha_n\}$ is in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $\beta_i \in (0, 1)$ *with* $\sum_{i=0}^N \beta_i = 1$.

Then, $\{x_n\}$ converges strongly to $\bar{z} \in \Gamma$ which solves the variational inequality

$$\langle \overrightarrow{\bar{z}h(\bar{z})}, \overrightarrow{u\bar{z}} \rangle \geq 0, \quad \forall u \in \Gamma. \tag{3.5}$$

Proof **Step 1** We first show that (3.4) is well defined. Let $S_{\lambda_n} x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \beta_2 J_{\lambda_n}^{A_2} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n$, then by Lemma 3.2, we have that S_{λ_n} is nonexpansive for all $n \geq 1$. Now, define the mapping $T_n^h : X \to X$ as follows:

$$T_n^h x = \alpha_n h(x) \oplus (1 - \alpha_n) T S_{\lambda_n} x.$$

Since T is nonexpansive, we obtain from Lemma 3.17 (iv) that

$$\begin{aligned} d(T_n^h x, T_n^h y) &\leq \alpha_n d(h(x), h(y)) + (1 - \alpha_n) d(T S_n x, T S_n y) \\ &\leq \tau \alpha_n d(x, y) + (1 - \alpha_n) d(S_n x, S_n y) \\ &\leq (\tau \alpha_n + (1 - \alpha_n))\, d(x, y). \end{aligned}$$

Since $\tau \in (0, 1)$, we have that $0 < (\tau\alpha_n + (1 - \alpha_n)) < 1$. Hence, T_n^h is a contraction for each $n \geq 1$. Therefore, by Banach contraction mapping principle, there exists a unique fixed point x_n of T_n^h for each $n \geq 1$. Thus, (3.4) is well defined.

Step 2 Next, we show that $\{x_n\}$ is bounded. Let $v \in \Gamma$, by (3.4) and Lemma 3.17 (i), we obtain

$$\begin{aligned} d(x_n, v) &\leq \alpha_n d(h(x_n), v) + (1 - \alpha_n) d(T y_n, v) \\ &\leq \alpha_n \tau d(x_n, v) + \alpha_n d(h(v), v) + (1 - \alpha_n) d(y_n, v) \\ &\leq \Big(1 - \alpha_n(1 - \tau)\Big) d(x_n, v) + \alpha_n d(h(v), v), \end{aligned} \tag{3.6}$$

which implies that

$$d(x_n, v) \leq \frac{d(h(v), v)}{1 - \tau}.$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$ $\{T y_n\}$ and $\{h(x_n)\}$ are all bounded.

Step 3 We now show that $\lim_{n\to\infty} d(x_n, T y_n) = \lim_{n\to\infty} d(x_n, T S_{\lambda_n} x_n) = 0$ and $\bar{z} \in \Gamma$.

From (3.4) and Lemma 2.10, we obtain that

$$\begin{aligned} d(x_n, Ty_n) &= d(\alpha_n h(x_n) \oplus (1-\alpha_n)Ty_n, Ty_n) \\ &\leq \alpha_n d(h(x_n), Ty_n). \end{aligned} \tag{3.7}$$

Since $\{h(x_n)\}$ and $\{Ty_n\}$ are bounded, we obtain from condition (i) and (3.7) that

$$\lim_{n\to\infty} d(x_n, Ty_n) = \lim_{n\to\infty} d(x_n, TS_{\lambda_n} x_n) = 0. \tag{3.8}$$

Now, by the boundedness of $\{x_n\}$ and the completeness of X, we obtain from Lemma 2.12 that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta - \lim_{k\to\infty} x_{n_k} = \bar{v}$. Again, since $T \circ S_{\lambda_n}$ is nonexpansive (and every nonexpansive mapping is semiclosed), it follows from (3.8), Lemma 3.2 and Theorem 2.8 (i) that $\bar{v} \in F(T \circ S_{\lambda_n}) \subseteq \bigcap_{i=1}^{N} F(J_\lambda^{A_i}) \cap F(T) = \Gamma$.

Step 4 We show that $\{x_n\}$ converges strongly to $\bar{z}$. Since $\{x_{n_k}\}$ Δ-converges to $\bar{v} \in \Gamma$, it follows from Lemma 2.13 that there exists $\bar{z} \in \Gamma$ such that $\{x_n\}$ Δ-converges to $\bar{z}$. Thus, by Lemma 2.14, we obtain that

$$\lim_{n\to\infty} \langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}} \rangle \leq 0. \tag{3.9}$$

Also, by Lemma 2.10 (iii) and (3.4), we have

$$\begin{aligned} d^2(x_n, \bar{z}) &\leq \alpha_n^2 d^2(h(x_n), \bar{z}) + (1-\alpha_n) d^2(Ty_n, \bar{z}) \\ &\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{h(x_n)\bar{z}}, \overrightarrow{Ty_n\bar{z}} \rangle \\ &\leq \alpha_n^2 d^2(h(x_n), \bar{z}) + (1-\alpha_n) d^2(x_n, \bar{z}) \\ &\quad + 2\alpha_n(1-\alpha_n)[\langle \overrightarrow{h(x_n)\bar{z}}, \overrightarrow{Ty_nx_n} \rangle + \langle \overrightarrow{h(x_n)h(\bar{z})}, \overrightarrow{x_n\bar{z}} \rangle \\ &\quad + \langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}} \rangle] \\ &\leq \alpha_n^2 d^2(h(x_n), \bar{z}) + (1-\alpha_n) d^2(x_n, \bar{z}) \\ &\quad + 2\alpha_n(1-\alpha_n)[\langle \overrightarrow{h(x_n)\bar{z}}, \overrightarrow{Ty_nx_n} \rangle + \tau d^2(x_n, \bar{z}) + \langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}} \rangle] \\ &\leq \Big[(1-\alpha_n) + 2\tau\alpha_n(1-\alpha_n)\Big] d^2(x_n, \bar{z}) \\ &\quad + \alpha_n\Big[\alpha_n d^2(h(x_n), \bar{z}) + 2(1-\alpha_n) d(Ty_n, x_n)\Big] d(h(x_n), \bar{z}) \\ &\quad + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}} \rangle. \end{aligned} \tag{3.10}$$

Therefore

$$d^2(x_n, \bar{z}) \leq \frac{[\alpha_n d^2(h(x_n), \bar{z}) + 2(1-\alpha_n) d(Ty_n, x_n)] d(h(x_n), \bar{z})}{[1 - 2\tau(1-\alpha_n)]} + \frac{2(1-\alpha_n)\langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}} \rangle}{[1 - 2\tau(1-\alpha_n)]},$$

which implies from condition (i), (3.8) and (3.9) that

$$\lim_{n\to\infty} d^2(x_n, \bar{z}) = 0.$$

Therefore, $\lim\limits_{n\to\infty} x_n = \bar{z}$.

Step 5 Finally, we show that $\bar{z}$ is a solution of (3.5). From Lemma 2.10 (ii) and (3.4), we obtain for all $u \in \Gamma$ that

$$\begin{aligned} d^2(x_m, u) &\leq \alpha_m d^2(h(x_m), u) + (1-\alpha_m) d^2(Ty_m, u) \\ &\quad - \alpha_m(1-\alpha_m) d^2(h(x_m), Ty_m) \end{aligned}$$

$$\leq \alpha_m d^2(h(x_m), u) + (1-\alpha_m)d(x_m, u) - \alpha_m(1-\alpha_m)d^2(h(x_m), Ty_m),$$

which implies

$$d^2(x_m, u) \leq d^2(h(x_m), u) - (1-\alpha_m)d^2(h(x_m), Ty_m).$$

Thus, taking limit as $m \to \infty$, we obtain

$$d^2(\bar{z}, u) \leq d^2(h(\bar{z}), u) - d^2(h(\bar{z}), \bar{z}).$$

Hence,

$$\langle \overrightarrow{\bar{z}h(\bar{z})}, \overrightarrow{u\bar{z}} \rangle = \frac{1}{2}\Big(d^2(h(\bar{z}), u) - d^2(\bar{z}, u) - d^2(h(\bar{z}), \bar{z})\Big) \geq 0, \quad \forall u \in \Gamma.$$

Therefore, we have that $\bar{z}$ solves the variational inequality (3.5).

Now, assume that $\{x_{n_k}\}$ $\triangle$-converges to u. Then, by the same argument, we obtain that $u \in \Gamma$ solves the variational inequality (3.5). That is,

$$\langle \overrightarrow{uh(u)}, \overrightarrow{u\bar{z}} \rangle \leq 0. \quad \text{Also} \quad \langle \overrightarrow{\bar{z}h(\bar{z})}, \overrightarrow{\bar{z}u} \rangle \leq 0.$$

Now, adding both, we get

$$\begin{aligned} 0 &\geq \langle \overrightarrow{\bar{z}h(\bar{z})}, \overrightarrow{\bar{z}u} \rangle - \langle \overrightarrow{uh(u)}, \overrightarrow{\bar{z}u} \rangle \\ &= \langle \overrightarrow{\bar{z}h(u)}, \overrightarrow{\bar{z}u} \rangle + \langle \overrightarrow{h(u)h(\bar{z})}, \overrightarrow{\bar{z}u} \rangle \\ &\quad - \langle \overrightarrow{u\bar{z}}, \overrightarrow{\bar{z}u} \rangle - \langle \overrightarrow{\bar{z}h(u)}, \overrightarrow{\bar{z}u} \rangle \\ &= \langle \overrightarrow{\bar{z}u}, \overrightarrow{\bar{z}u} \rangle - \langle \overrightarrow{h(u)h(\bar{z})}, \overrightarrow{u\bar{z}} \rangle \\ &\geq \langle \overrightarrow{\bar{z}u}, \overrightarrow{\bar{z}u} \rangle - d(h(u)h(\bar{z}))d(u, \bar{z}) \\ &\geq d^2(\bar{z}, u) - \tau d^2(u, \bar{z}) \\ &= (1-\tau)d^2(\bar{z}, u). \end{aligned}$$

which implies that $d(\bar{z}, u) = 0$. Hence, $\bar{z} = u$. Therefore, $\{x_n\}$ converges strongly to $\bar{z}$, which is a solution of the variational inequality (3.5). □

By setting $T \equiv I$ (where I is the identity mapping on X) and $h(x) = c$ for arbitrary but fixed $c \in X$ and $\forall x \in X$, we obtain the following corollary.

Corollary 3.4 *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be a finite family of multivalued monotone operators that satisfy the range condition. Suppose that $\Gamma := \bigcap_{i=1}^N A_i^{-1}(0) \neq \emptyset$ and for arbitrary $c,\ x_1 \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \beta_2 J_{\lambda_n}^{A_2} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n, \\ x_n = \alpha_n c \oplus (1-\alpha_n) y_n, \quad n \geq 1, \end{cases} \tag{3.11}$$

where $0 < \lambda \leq \lambda_n$ $\forall n \geq 1$ and $\{\alpha_n\}$ is in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $\beta_i \in (0, 1)$ *with* $\sum_{i=0}^{N} \beta_i = 1$.

Then, $\{x_n\}$ converges strongly to $\bar{z} \in \Gamma$.

By setting $N = 1$ in Theorem 3.3, we obtain the following result.

Corollary 3.5 *Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be a multivalued monotone operator that satisfies the range condition. Let T be a nonexpansive mapping on X and h be a contraction mapping on X with coefficient $\tau \in (0, 1)$. Suppose that $\Gamma := F(T) \cap A^{-1}(0) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is generated by*

$$x_n = \alpha_n h(x_n) \oplus (1-\alpha_n)T\left(\beta_0 x_n \oplus \beta_1 J_{\lambda_n}^A x_n\right), \quad n \geq 1, \tag{3.12}$$

where $0 < \lambda \leq \lambda_n$ $\forall n \geq 1$ and $\{\alpha_n\}$ is in $(0, 1)$ satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(ii) $\beta_i \in (0, 1),\ i = 0, 1$ *with* $\beta_0 + \beta_1 = 1$.

Then, $\{x_n\}$ converges strongly to $\bar{z} \in \Gamma$ which solves the variational inequality

$$\langle \overrightarrow{\bar{z}h(\bar{z})}, \overrightarrow{u\bar{z}} \rangle \geq 0, \quad \forall u \in \Gamma. \tag{3.13}$$

The following remark will be needed in the proof of the next theorem.

Remark 3.6 If X is a CAT(0) space and $A : X \to 2^{X^*}$ is a multivalued monotone mapping, then for $0 < \lambda \leq \mu$, we have that

$$d(J_\lambda^A x, J_\mu^A x) \leq \left(\sqrt{1-\frac{\lambda}{\mu}}\right) d(x, J_\mu^A x), \quad \forall x \in X.$$

Indeed, from Theorem 2.8 (iii), we obtain that

$$\frac{\mu+\lambda}{\mu} d^2(J_\lambda^A x, J_\mu^A x) \leq \frac{\mu-\lambda}{\mu} d^2(x, J_\mu^A x),$$

which implies that

$$d^2(J_\lambda^A x, J_\mu^A x) \leq \left(1-\frac{\lambda}{\mu}\right) d^2(x, J_\mu^A x).$$

That is,

$$d(J_\lambda^A x, J_\mu^A x) \leq \left(\sqrt{1-\frac{\lambda}{\mu}}\right) d(x, J_\mu^A x).$$

Theorem 3.7 *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*},\ i = 1, 2, \ldots, N$ be a finite family of multivalued monotone operators that satisfy the range condition. Let T be a nonexpansive mapping on X and h be a contraction mapping on X with coefficient $\tau \in (0, 1)$. Suppose that $\Gamma := F(T) \cap \left(\bigcap_{i=1}^N A_i^{-1}(0)\right) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n, \\ w_n = \frac{\alpha_n}{1-\beta_n} h(x_n) \oplus \frac{\gamma_n}{1-\beta_n} T y_n, \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) w_n, \quad n \geq 1. \end{cases} \tag{3.14}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1\ \forall n \ge 1$,
(iii) $0 < \lambda \le \lambda_n\ \forall n \ge 1$ *and* $\lim_{n\to\infty}\lambda_n = \lambda$,
(iv) $\beta_i \in (0, 1)$ *with* $\sum_{i=0}^{N}\beta_i = 1$.

Then, $\{x_n\}$ *converges strongly to* $\bar{z} \in \Gamma$.

Proof **Step 1** We show that $\{x_n\}$ is bounded. Let $u \in \Gamma$ and set $S_{\lambda_n}x_n := \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1}x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N}x_n$, then from (3.14), Lemma 2.10 (i) and Lemma 3.2, we obtain that

$$\begin{aligned}
d(x_{n+1}, u) &\le \beta_n d(x_n, u) + (1-\beta_n)d(w_n, u)\\
&\le \beta_n d(x_n, u) + (1-\beta_n)\left[\frac{\alpha_n}{1-\beta_n}d(h(x_n), u) + \frac{\gamma_n}{1-\beta_n}d(Ty_n, u)\right]\\
&\le \beta_n d(x_n, u) + (1-\beta_n)\left[\frac{\alpha_n}{1-\beta_n}\tau d(x_n, u) + \frac{\alpha_n}{1-\beta_n}d(h(u), u)\right.\\
&\quad \left. + \frac{\gamma_n}{1-\beta_n}d(Ty_n, u)\right]\\
&\le (\beta_n + \tau\alpha_n)d(x_n, u) + \gamma_n d(S_{\lambda_n}x_n, u) + \alpha_n d(h(u), u)\\
&= (1-\alpha_n(1-\tau))d(x_n, u) + \alpha_n d(h(u), u)\\
&\le \max\left\{d(x_n, u) + \frac{d(h(u), u)}{1-\tau}\right\}\\
&\ \ \vdots\\
&\le \max\left\{d(x_1, u) + \frac{d(h(u), u)}{1-\tau}\right\}.
\end{aligned}$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{y_n\}$, $\{h(x_n)\}$ and $\{T(y_n)\}$ are all bounded.

Step 2 Next, we show that $\lim_{n\to\infty} d(x_{n+1}, x_n) = 0$. Now, from (3.14), Lemma 2.10 (iv),(vi) and the nonexpansivity of T, we obtain that

$$\begin{aligned}
d(w_{n+1}, w_n) &= d\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}}h(x_{n+1}) \oplus \frac{\gamma_{n+1}}{1-\beta_{n+1}}Ty_{n+1}, \frac{\alpha_n}{1-\beta_n}h(x_n) \oplus \frac{\gamma_n}{1-\beta_n}Ty_n\Big)\\
&\le d\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}}h(x_{n+1}) \oplus \big(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\big)Ty_{n+1}, \frac{\alpha_{n+1}}{1-\beta_{n+1}}h(x_n) \oplus \big(1\\
&\quad -\frac{\alpha_{n+1}}{1-\beta_{n+1}}\big)Ty_n\Big)\\
&\quad + d\Big(\frac{\alpha_{n+1}}{1-\beta_{n+1}}h(x_n) \oplus \big(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\big)Ty_n, \frac{\alpha_n}{1-\beta_n}h(x_n) \oplus \big(1\\
&\quad -\frac{\alpha_n}{1-\beta_n}\big)Ty_n\Big)\\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\tau d(x_{n+1}, x_n) + \big(1-\frac{\alpha_{n+1}}{1-\beta_{n+1}}\big)d(y_{n+1}, y_n)\\
&\quad + \left|\frac{\alpha_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n}{1-\beta_n}\right| d(h(x_n), Ty_n) \qquad (3.15)
\end{aligned}$$

Without loss of generality, we may assume that $0 < \lambda_{n+1} \le \lambda_n\ \forall n \ge 1$. Thus, from (3.14), condition (iv), Lemma 3.1 and Remark 3.6, we obtain

$$
\begin{aligned}
d(y_{n+1}, y_n) &= d\big(\beta_0 x_{n+1} \oplus \beta_1 J_{\lambda_{n+1}}^{A_1} x_{n+1} \oplus \cdots \oplus \beta_N J_{\lambda_{n+1}}^{A_N} x_{n+1},\ \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \cdots \\
&\quad \oplus \beta_N J_{\lambda_n}^{A_N} x_n\big) \\
&\le \beta_0 d(x_{n+1}, x_n) + \sum_{i=1}^{N} \beta_i d(J_{\lambda_{n+1}}^{A_i} x_{n+1}, J_{\lambda_n}^{A_i} x_n) \\
&\le \beta_0 d(x_{n+1}, x_n) + \sum_{i=1}^{N} \beta_i d(J_{\lambda_{n+1}}^{A_i} x_{n+1}, J_{\lambda_{n+1}}^{A_i} x_n) + \sum_{i=1}^{N} \beta_i d(J_{\lambda_{n+1}}^{A_i} x_n, J_{\lambda_n}^{A_i} x_n) \\
&\le d(x_{n+1}, x_n) + \left(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\right) \sum_{i=1}^{N} \beta_i d(J_{\lambda_n}^{A_i} x_n, x_n) \\
&\le d(x_{n+1}, x_n) + \left(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\right) M,
\end{aligned} \tag{3.16}
$$

where $M := \sup_{n \ge 1} \left\{ \sum_{i=1}^{N} \beta_i d(J_{\lambda_n}^{A_i} x_n, x_n) \right\}$. Substituting (3.16) into (3.15), we obtain that

$$
\begin{aligned}
d(w_{n+1}, w_n) &\le \frac{\alpha_{n+1}}{1 - \beta_{n+1}} \tau d(x_{n+1}, x_n) + \left(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\right) d(x_{n+1}, x_n) \\
&\quad + \left(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\right)\left(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\right) M \\
&\quad + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| d(h(x_n), T y_n) \\
&= \left[1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} (1 - \tau)\right] d(x_{n+1}, x_n) + \left(\sqrt{1 - \frac{\lambda_{n+1}}{\lambda_n}}\right)\left(1 - \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\right) M \\
&\quad + \left| \frac{\alpha_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n}{1 - \beta_n} \right| d(h(x_n), T y_n).
\end{aligned}
$$

Since $\lim_{n \to \infty} \alpha_n = 0$, $\quad \lim_{n \to \infty} \lambda_n = \lambda$ and $\{h(x_n)\}$, $\{T y_n\}$ are bounded, we obtain that

$$
\limsup_{n \to \infty} \big(d(w_{n+1}, w_n) - d(x_{n+1}, x_n)\big) \le 0.
$$

Thus, by Lemma 2.15 and condition (ii), we obtain that

$$
\lim_{n \to \infty} d(w_n, x_n) = 0. \tag{3.17}
$$

Hence, by Lemma 2.10, we obtain that

$$
d(x_{n+1}, x_n) \le (1 - \beta_n) d(w_n, x_n) \to 0, \quad \text{as } n \to \infty. \tag{3.18}
$$

Step 3 We now show that $\lim_{n \to \infty} d(x_n, T(S_{\lambda_n}) x_n) = 0 = \lim_{n \to \infty} d(w_n, T(S_{\lambda_n}) w_n)$. Observe from Remark 2.17 that (3.14) can be rewritten as

$$
\begin{cases}
y_n = \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N} x_n, \\
x_{n+1} = \alpha_n h(x_n) \oplus (1 - \alpha_n) \left(\frac{\beta_n x_n \oplus \gamma_n T y_n}{(1 - \alpha_n)} \right), \quad \ge 1.
\end{cases} \tag{3.19}
$$

Thus, by Lemma 2.10, we obtain that

$$
d\left(x_{n+1}, \frac{\beta_n x_n \oplus \gamma_n T y_n}{(1 - \alpha_n)}\right) \le \alpha_n d\left(h(x_n), \frac{\beta_n x_n \oplus \gamma_n T y_n}{(1 - \alpha_n)}\right) \to 0, \quad \text{as } n \to \infty. \tag{3.20}
$$

Also, from (2.1), we obtain

$$d\Big(x_n, \frac{\beta_n x_n \oplus \gamma_n T y_n}{(1-\alpha_n)}\Big) = \frac{\gamma_n}{1-\alpha_n} d(x_n, Ty_n),$$

which implies from (3.18) and (3.20) that

$$\frac{\gamma_n}{1-\alpha_n} d(x_n, Ty_n) \le d(x_n, x_{n+1}) + d\Big(x_{n+1}, \frac{\beta_n x_n \oplus \gamma_n T y_n}{(1-\alpha_n)}\Big) \to 0, \quad \text{as } n \to \infty.$$

Hence,

$$\lim_{n\to\infty} d(x_n, Ty_n) = \lim_{n\to\infty} d(x_n, T(S_{\lambda_n})x_n) = 0. \tag{3.21}$$

Since $\{x_n\}$ is bounded and X is an Hadamard space, then by Lemma 2.12, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\Delta - \lim_{k\to\infty} x_{n_k} = \bar{u}$. Again, by the nonexpansivity of $T \circ S_{\lambda_n}$, we obtain from (3.21), condition (iii), Lemma 3.2 and Lemma 2.8 (i) that $\bar{u} \in F(T \circ S_{\lambda_n}) \subseteq \bigcap_{i=1}^N F(J_\lambda^{A_i}) \cap F(T) = \Gamma$.

Also, by (3.17) and (3.21), we obtain

$$\begin{aligned} d(w_n, T(S_{\lambda_n})w_n) &\le d(w_n, x_n) + d(x_n, T(S_{\lambda_n})x_n) + d(T(S_{\lambda_n})x_n, T(S_{\lambda_n})w_n) \\ &\le 2d(w_n, x_n) + d(x_n, T(S_{\lambda_n}x_n) \to 0, \text{ as } n \to \infty. \end{aligned} \tag{3.22}$$

Step 4 Next, we show that $\limsup_{n\to\infty} \langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}} \rangle \le 0$.
If we set $T_m^h x := \beta_m x \oplus (1-\beta_m)w$, where $w = \frac{\alpha_m}{(1-\beta_m)} h(x) \oplus \frac{\gamma_m}{(1-\beta_m)} T(S_{\lambda_m})x$, then by following the same method of proof as in the proof of Theorem 3.3, we get that T_m^h is a contraction for each $m \ge 1$. Thus, there exists a unique fixed point z_m of T_m^h $\forall m \ge 1$. That is,

$z_m = \beta_m z_m \oplus (1-\beta_m)w_m$, where $w_m = \frac{\alpha_m}{(1-\beta_m)} h(z_m) \oplus \frac{\gamma_m}{(1-\beta_m)} T(S_{\lambda_m})z_m$. Furthermore, it follows from Theorem 3.3 that $\lim_{m\to\infty} z_m = \bar{z} \in \Gamma$. Thus, we obtain that

$$\begin{aligned} d(z_m, w_n) &= d(\beta_m z_m \oplus (1-\beta_m)w_m, w_n) \\ &\le \beta_m d(z_m, w_n) + (1-\beta_m)d(w_m, w_n), \end{aligned}$$

which implies that

$$d(z_m, w_n) \le d(w_m, w_n). \tag{3.23}$$

From (3.23) and Lemma 2.10(v), we obtain that

$$\begin{aligned} d^2(w_m, w_n) &= \langle \overrightarrow{w_m w_n}, \overrightarrow{w_m w_n} \rangle \\ &= \langle \overrightarrow{w_m T(S_{\lambda_m})z_m}, \overrightarrow{w_m w_n} \rangle + \langle \overrightarrow{T(S_{\lambda_m})z_m w_n}, \overrightarrow{w_m w_n} \rangle \\ &\le \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{h(z_m)T(S_{\lambda_m})z_m}, \overrightarrow{w_m w_n} \rangle + \langle \overrightarrow{T(S_{\lambda_m} z_m)w_n}, \overrightarrow{w_m w_n} \rangle \\ &= \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{h(z_m)T(S_{\lambda_m} z_m)}, \overrightarrow{w_m z_m} \rangle + \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{h(z_m)w_n}, \overrightarrow{z_m w_n} \rangle \\ &\quad + \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{w_n T(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_m} \rangle + \langle \overrightarrow{T(S_{\lambda_m} z_m)T(S_{\lambda_m} w_n)}, \overrightarrow{w_m w_m} \rangle \\ &\quad + \langle \overrightarrow{T(S_{\lambda_m} w_m)}, \overrightarrow{w_m w_n} \rangle \\ &\le \frac{\alpha_m}{(1-\beta_m)} d(h(z_m), T(S_{\lambda_m} z_m)) d(w_m, z_m) + \frac{\alpha_m}{(1-\beta_m)} \langle \overrightarrow{h(z_m)z_m}, \overrightarrow{z_m w_n} \rangle \end{aligned}$$

$$
\begin{aligned}
&+ \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{z_m T(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_n}\rangle + d(T(S_{\lambda_m} z_m), T(S_{\lambda_m} w_n))d(w_m, w_n)\\
&+ d(T(S_{\lambda_m} w_n), w_n)d(w_m, w_n)\\
&\le \frac{\alpha_m}{(1-\beta_m)} d(h(z_m), T(S_{\lambda_m} z_m))d(w_n, z_m) + \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{h(z_m)z_m}, \overrightarrow{z_m w_n}\rangle\\
&+ \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{z_m T(S_{\lambda_m} z_m)}, \overrightarrow{z_m w_n}\rangle + d(z_m, w_m)d(w_m, w_n)\\
&+ d(T(S_{\lambda_m} w_n), w_n)d(w_n, w_m)\\
&\le \frac{\alpha_m}{(1-\beta_m)} d(h(z_m), T(S_{\lambda_m} z_m))d(w_n, z_m) + \frac{\alpha_m}{(1-\beta_m)}\langle \overrightarrow{h(z_m)z_m}, \overrightarrow{z_m w_n}\rangle\\
&+ \frac{\alpha_m}{(1-\beta_m)} d(z_m, T(S_{\lambda_m} z_m))d(w_m, z_m) + d(w_m, w_n)\\
&+ d(T(S_{\lambda_m} w_n), w_n)d(w_n, w_m),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\langle \overrightarrow{h(z_m)z_m}, \overrightarrow{w_n z_m}\rangle \le\ & d(h(z_m), T(S_{\lambda_m})z_m)d(w_n, z_m) + d(z_m, T(S_{\lambda_m})z_m)d(z_m, w_m)\\
&+ \frac{(1-\beta_m)}{\alpha_m} d(T(S_{\lambda_n})w_n, w_n)d(w_m, w_m).
\end{aligned}
$$

Thus, taking lim sup as $n \to \infty$ first, then as $m \to \infty$, it follows from (3.17),(3.21) and (3.22) that

$$
\limsup_{m\to\infty}\limsup_{n\to\infty}\langle \overrightarrow{h(z_m)z_m}, \overrightarrow{w_n z_m}\rangle \le 0. \tag{3.24}
$$

Furthermore,

$$
\begin{aligned}
\langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle &= \langle \overrightarrow{h(\bar{z})h(z_m)}, \overrightarrow{x_n\bar{z}}\rangle + \langle \overrightarrow{h(z_m)z_m}, \overrightarrow{x_n w_n}\rangle + \langle \overrightarrow{h(z_m)z_m}, \overrightarrow{w_n z_m}\rangle\\
&\quad + \langle \overrightarrow{h(z_m)z_m}, \overrightarrow{z_m\bar{z}}\rangle + \langle \overrightarrow{z_m\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle\\
&\le d(h(\bar{z}), h(z_m))d(x_n, \bar{z}) + d(h(z_m), z_m)d(x_n, w_n) + \langle \overrightarrow{h(z_m)z_m}, \overrightarrow{w_n z_m}\rangle\\
&\quad + d(h(z_m), z_m)d(z_m, \bar{z}) + d(z_m, \bar{z})d(x_n, \bar{z})\\
&\le (1+\tau)d(z_m, \bar{z})d(x_n, \bar{z}) + \langle \overrightarrow{h(z_m)z_m}, \overrightarrow{w_n z_m}\rangle\\
&\quad + [d(x_n, w_n) + d(z_m, \bar{z})]d(h(z_m), z_m),
\end{aligned}
$$

which implies from (3.17), (3.24) and the fact that $\lim_{m\to\infty} z_m = \bar{z}$, that

$$
\begin{aligned}
\limsup_{n\to\infty}\langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle &= \limsup_{m\to\infty}\limsup_{n\to\infty}\langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle\\
&\le \limsup_{m\to\infty}\limsup_{n\to\infty}\langle \overrightarrow{h(z_m)z_m}, \overrightarrow{w_n z_m}\rangle \le 0.
\end{aligned} \tag{3.25}
$$

Step 5 Finally, we show that $\{x_n\}$ converges strongly to $\bar{z} \in \Gamma$.

From Lemma 2.11, we obtain that

$$
\begin{aligned}
\langle \overrightarrow{w_n\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle &\le \frac{\alpha_n}{(1-\beta_n)}\langle \overrightarrow{h(x_n)\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle + \frac{\gamma_n}{(1-\beta_n)}\langle \overrightarrow{T(S_{\lambda_n})x_n\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle\\
&\le \frac{\alpha_n}{(1-\beta_n)}\langle \overrightarrow{h(x_n)h(\bar{z})}, \overrightarrow{x_n\bar{z}}\rangle + \frac{\alpha_n}{(1-\beta_n)}\langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle\\
&+ \frac{\gamma_n}{(1-\beta_n)} d(T(S_{\lambda_n})x_n, \bar{z})d(x_n, \bar{z})
\end{aligned}
$$

$$\leq \frac{\alpha_n}{(1-\beta_n)}\tau d^2(x_n,\bar{z}) + \frac{\alpha_n}{(1-\beta_n)}\langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle + \left(1-\frac{\alpha_n}{1-\beta_n}\right)d^2(x_n,\bar{z})$$
$$= \left[\frac{\alpha_n}{(1-\beta_n)}\tau + \left(1-\frac{\alpha_n}{1-\beta_n}\right)\right]d^2(x_n,\bar{z}) + \frac{\alpha_n}{(1-\beta_n)}\langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle.$$

Thus, from Lemma 2.10, we have

$$\begin{aligned} d^2(x_{n+1},\bar{z}) &\leq \beta_n d^2(x_n,\bar{z}) + (1-\beta_n)d^2(w_n,\bar{z}) \\ &= \beta_n d^2(x_n,\bar{z}) + (1-\beta_n)\langle \overrightarrow{w_n\bar{z}}, \overrightarrow{w_n\bar{z}}\rangle \\ &= \beta_n d^2(x_n,\bar{z}) + (1-\beta_n)[\langle \overrightarrow{w_n\bar{z}}, \overrightarrow{w_nx_n}\rangle + \langle \overrightarrow{w_n\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle] \\ &\leq [\beta_n + \alpha_n\tau + \gamma_n]d^2(x_n,\bar{z}) + (1-\beta_n)\langle \overrightarrow{w_n\bar{z}}, \overrightarrow{w_nx_n}\rangle + \alpha_n\langle h(\bar{z})\bar{z}, x_n\bar{z}\rangle \\ &\leq (1-\alpha_n(1-\tau))d^2(x_n,\bar{z}) + \alpha_n(1-\tau)\left[\frac{1}{1-\tau}\langle \overrightarrow{h(\bar{z})\bar{z}}, \overrightarrow{x_n\bar{z}}\rangle\right] \\ &\quad + (1-\beta_n)d(w_n,x_n)d(w_n,\bar{z}). \end{aligned} \tag{3.26}$$

By (3.17) and applying Lemma 2.16 to (3.26), we obtain that $\{x_n\}$ converges strongly to $\bar{z}$. □

By setting $T \equiv I$ in Theorem 3.7, where I is an identity mapping on X, we obtain the following result.

Corollary 3.8 *Let X be an Hadamard space and X^* be its dual space. Let $A_i : X \to 2^{X^*}$, $i = 1, 2, \ldots, N$ be a finite family of multivalued monotone operators that satisfy the range condition. Let h be a contraction mapping on X with coefficient $\tau \in (0, 1)$. Suppose that $\Gamma := \bigcap_{i=1}^N A_i^{-1}(0) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A_1}x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{A_N}x_n, \\ w_n = \frac{\alpha_n}{1-\beta_n}h(x_n) \oplus \frac{\gamma_n}{1-\beta_n}y_n, \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n)w_n, \quad n \geq 1. \end{cases} \tag{3.27}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^\infty \alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1\ \forall n \geq 1$,
(iii) $0 < \lambda \leq \lambda_n\ \forall n \geq 1$ *and* $\lim_{n\to\infty}\lambda_n = \lambda$,
(iv) $\beta_i \in (0, 1)$ *with* $\sum_{i=0}^N \beta_i = 1$.

Then, $\{x_n\}$ converges strongly to $\bar{z} \in \Gamma$.

By setting $N = 1$ in Theorem 3.7, we obtain the following result.

Corollary 3.9 *Let X be an Hadamard space and X^* be its dual space. Let $A : X \to 2^{X^*}$ be a multivalued monotone operator that satisfies the range condition. Let T be a nonexpansive mapping on X and h be a contraction mapping on X with coefficient $\tau \in (0, 1)$. Suppose that $\Gamma := A^{-1}(0) \cap F(T) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} w_n = \frac{\alpha_n}{1-\beta_n}h(x_n) \oplus \frac{\gamma_n}{1-\beta_n}T\left(\beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{A}x_n\right), \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n)w_n, \quad n \geq 1. \end{cases} \tag{3.28}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1\ \forall n \geq 1$,
(iii) $0 < \lambda \leq \lambda_n\ \forall n \geq 1$ *and* $\lim_{n\to\infty}\lambda_n = \lambda$,
(iv) $\beta_i \in (0, 1)$, $i = 0, 1$ *with* $\beta_0 + \beta_1 = 1$.

Then, $\{x_n\}$ *converges strongly to* $\bar{z} \in \Gamma$.

4 Applications to some optimization problems

In this section, we apply our results to solve some optimization problems. Throughout this section, we assume that X is an Hadamard space, X^* is its dual space and $f : X \to (-\infty, \infty]$ is a proper convex and lower semicontinuous function with domain $D(f) := \{x \in X : f(x) < +\infty\}$. Recall that a function $f : X \to (-\infty, \infty]$ is called

(i) *convex*, if

$$f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)\ \forall x, y \in X,\ \lambda \in (0, 1),$$

(ii) *proper,* if $D(f) \neq \emptyset$,
(iii) *lower semi-continuous at a point* $x \in D(f)$, if

$$f(x) \leq \liminf_{n\to\infty} f(x_n),\ \text{for each sequence } \{x_n\} \text{ in } D(f) \text{ such that } \lim_{n\to\infty} x_n = x.$$

Moreover, f is said to be lower semicontinuous on $D(f)$, if it is lower semicontinuous at any point in $D(f)$.

The subdifferential $\partial f : X \to 2^{X^*}$ of f, defined by

$$\partial f(x) = \begin{cases} \{x^* \in X^* : f(z) - f(x) \geq \langle x^*, \overrightarrow{xz}\rangle,\ \forall z \in X\}, & \text{if } x \in D(f), \\ \emptyset, & \text{otherwise} \end{cases} \tag{4.1}$$

is (see [1])

(i) a monotone operator,
(ii) known to satisfy the range condition. That is, $D(J_\lambda^{\partial f}) = X$ for all $\lambda > 0$.

Furthermore, for a nonempty, closed and convex subset C of X. The indicator function $\delta_C : X \to \mathbb{R}$ defined by

$$\delta_C(x) = \begin{cases} 0, & \text{if } x \in C, \\ +\infty, & \text{otherwise} \end{cases} \tag{4.2}$$

is a proper convex and lower semicontinuous function. Thus, the subdifferential of δ_C,

$$\partial\delta_C(x) = \begin{cases} \{x^* \in X^* : \langle x^*, \overrightarrow{xz}\rangle \leq 0\ \forall z \in C\} \text{ if } x \in C, & \\ \emptyset, & \text{otherwise} \end{cases} \tag{4.3}$$

is a monotone operator which satisfies the range condition.

4.1 Application to variational inequality problem

Let us consider the following variational inequality problem associated with a nonexpansive mapping T which was recently formulated in an Hadamard space by Khatibzadeh and Ranjbar [29]:

$$\text{Find } x \in C \text{ such that } \langle \overrightarrow{Txx}, \overrightarrow{xy} \rangle \geq 0 \quad \forall y \in C. \tag{4.4}$$

Recall that the metric projection $P_C : X \to C$ is defined for $x \in X$ by $d(x, P_C x) = \inf_{y \in C} d(x, y)$ and characterized by, $z = P_C x$ if and only if $\langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \leq 0, \ \forall y \in C$ (see [29]). Now, using the characterization of P_C, we obtain that

$$x = P_C T x \iff \langle \overrightarrow{Txx}, \overrightarrow{xy} \rangle \geq 0 \ \forall y \in C.$$

Therefore, $x \in F(P_C \circ T)$ if and only if x solves (4.4). Also, by (2.4) and (4.3), we obtain that

$$z = J_\lambda^{\partial \delta_C} x \iff [\frac{1}{\lambda} \overrightarrow{zx}] \in \partial \delta_C z \iff \langle \overrightarrow{zx}, \overrightarrow{zy} \rangle \leq 0, \ \forall \ y \in C \iff z = P_C x. \tag{4.5}$$

Thus, by letting $z = x$, we obtain that $x = P_C x$ if and only if $x \in (\partial \delta_C)^{-1}(0)$. Therefore, we get that

$$x \in (\partial \delta_C)^{-1}(0) \cap F(T) \implies x \in F(P_C) \cap F(T) \implies x \in F(P_C \circ T).$$

Thus, suppose that the solution set of problem (4.4) is Ω, then by setting $A = \partial \delta_C$ in Corollary 3.9, we apply Corollary 3.9 to obtain the following result for approximating solutions of variational inequality problem in an Hadamard space.

Theorem 4.1 *Let C be a nonempty closed and convex subset of an Hadamard space X and X^* be the dual space of X. Let $T : X \to X$ be a nonexpansive mapping and h be a contraction mapping on X with constant $\tau \in (0, 1)$. Suppose that $\Omega \neq \emptyset$ and the sequence $\{x_n\}$ is generated for arbitrary $x_1 \in X$ by*

$$\begin{cases} w_n = \frac{\alpha_n}{1-\beta_n} h(x_n) \oplus \frac{\gamma_n}{1-\beta_n} T \left(\beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{\partial \delta_C} x_n \right), \\ x_{n+1} = \beta_n x_n \oplus (1 - \beta_n) w_n, \quad n \geq 1. \end{cases} \tag{4.6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1 \ \forall n \geq 1$,
(iii) $0 < \lambda \leq \lambda_n \ \forall n \geq 1$ *and* $\lim_{n\to\infty} \lambda_n = \lambda$,
(iv) $\beta_i \in (0, 1), i = 0, 1$ *with* $\beta_0 + \beta_1 = 1$.

Then, $\{x_n\}$ converges strongly to an element of Ω.

4.2 Application to finite family of minimization problems

Consider the following Minimization Problem (MP): Find $x \in X$ such that

$$f(x) = \min_{y \in X} f(y). \tag{4.7}$$

It was proved in [1] that f attains its minimum at $x \in X$ if and only if $0 \in \partial f(x)$. Thus, the above MP (4.7) can be formulated as follows: Find $x \in X$ such that

$$0 \in \partial f(x).$$

Therefore, by setting $A = \partial f$ in Theorem 3.7, we obtain the following result.

Theorem 4.2 *Let X be an Hadamard space and X^* be its dual space. Let $f_i : X \to (-\infty, \infty]$, $i = 1, 2, \ldots, N$ be a finite family of proper, convex and lower semicontinuous functions. Let T be a nonexpansive mapping on X and h be a contraction mapping on X with constant $\tau \in (0, 1)$. Suppose that $\Gamma^* := F(T) \cap \left(\bigcap_{i=1}^{N} \partial f_i^{-1}(0)\right) \neq \emptyset$ and for arbitrary $x_1 \in X$, the sequence $\{x_n\}$ is generated by*

$$\begin{cases} y_n = \beta_0 x_n \oplus \beta_1 J_{\lambda_n}^{\partial f_1} x_n \oplus \cdots \oplus \beta_N J_{\lambda_n}^{\partial f_N} x_n, \\ w_n = \frac{\alpha_n}{1-\beta_n} h(x_n) \oplus \frac{\gamma_n}{1-\beta_n} T y_n, \\ x_{n+1} = \beta_n x_n \oplus (1-\beta_n) w_n, \quad n \geq 1. \end{cases} \tag{4.8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are sequences in $(0, 1)$, and $\{\lambda_n\}$ is a sequence of positive real numbers satisfying the following conditions:

(i) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,
(ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$, $\alpha_n + \beta_n + \gamma_n = 1 \ \forall n \geq 1$,
(iii) $0 < \lambda \leq \lambda_n \ \forall n \geq 1$ *and* $\lim_{n\to\infty} \lambda_n = \lambda$,
(iv) $\beta_i \in (0, 1)$ *with* $\sum_{i=0}^{N} \beta_i = 1$.

Then, $\{x_n\}$ converges strongly to $\bar{z} \in \Gamma^*$.

4.3 Convex feasibility problem

Let C be a nonempty closed and convex subset of X and $C_i, i = 1, 2, \ldots, N$ be a finite family of nonempty closed and convex subsets of C such that $\bigcap_{i=1}^{N} C_i \neq \emptyset$. The convex feasibility problem is defined as:

$$\text{Find } x \in C \text{ such that } x \in \bigcap_{i=1}^{N} C_i. \tag{4.9}$$

Now, observe that (4.5) implies that $x = J_{\lambda}^{\partial \delta_{C_i}} x \iff x = P_{C_i} x$, $i = 1, 2, \ldots, N$. Therefore, by setting $A_i = \partial \delta_{C_i}$ in Corollary 3.8 and $J_{\lambda_n}^{A_i} = P_{C_i}$, $i = 1, 2, \ldots, N$ in Algorithm (3.27), we can apply Corollary 3.8 to approximate solutions of (4.9).

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Compliance with ethical standards

Conflicts of interest The authors declare that they have no competing interests.

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